A Constructive Semantics for Classical Logic

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Background. *Constructivism* in logic and mathematics holds that the meaning of a proposition is given by what counts as a proof or verification of it, rather than by potentially verification-transcendent truth conditions. The Brouwer–Heyting–Kolmogorov (BHK) interpretation is the canonical expression of this position, explaining each logical constant in terms of what constitutes a proof of a compound proposition. Schroeder-Heister [1] argues that one of the ways of reifying this interpretation is *proof-theoretic semantics* (P-tS): it systematises the constructive programme by giving a precise, rule-based account of meaning in terms of inference.

The philosophical underpinning of P-tS is the Dummettian thesis that meaning is justification-practices. On this view, to grasp the meaning of a proposition is to know the conditions under which it is verified — what Dummett calls its *assertion conditions* — rather than the conditions under which it would be true independently of our capacity to recognise this. The logical constants receive their meaning from the rules governing their correct use in inference, not from their interpretation in a model. It is within this paradigm that the present paper gives a constructive account of *full classical propositional logic*.

The specific framework we employ is *base-extension semantics* (B-eS), in the tradition of Sandqvist [17, 18] and Piecha et al. [13, 14]. A *base* $\mathfrak{B}$ is a set of atomic inference rules encoding pre-logical inferential knowledge; the meaning of the logical constants is given by a *support* relation $\Gamma \Vdash_{\mathfrak{B}} \phi$ defined inductively on formulae, with the base case tied to derivability in $\mathfrak{B}$. Sandqvist [17] gave a B-eS for classical logic, but his completeness proof relies on bar induction; while it has been refined by Gheorghiu [5] to be elementary the technique does not extend to the full standard language. By contrast, Sandqvist's B-eS for *intuitionistic* logic [18] admits a strikingly elementary completeness proof, one that transfers to other logics with suitable natural deduction symmetry [6, 7]. Extending this constructive approach to classical logic with the full language — including $\vee$ as primitive — has remained an open problem [5].

Main Contribution. This paper presents a B-eS for classical propositional logic that resolves the above difficulties, built around the slogan:

Classical Logic $=$ Intuitionistic Logic $+$ Duality.

The key innovation is to operate over *literals* rather than atoms. Following speech-act theory [16, 19], but crucially *restricting* bilateralism to the atomic level, we distinguish a set $\mathcal{C}$ of *contents* from the propositions built from them. A *literal* is either a positive proposition c^+ (assertion of c) or its primitive dual c^- (denial of c); complex formulae built from literals by $\wedge, \vee, \to, \bot, \top$ are logico-grammatical structures, not themselves objects of illocutionary force. This restriction avoids the complications of uniform bilateralism while enabling the elementary completeness argument.

The System $\mathsf{NK}^{\pm}$. The natural deduction system $\mathsf{NK}^{\pm}$ consists of the standard rules of intuitionistic propositional logic NJ (operating over literals as atoms), augmented by two rules governing literal duality:

$$\frac{l \quad l^{\perp}}{\bot}\,\mathsf{Exc}_1 \qquad\qquad \frac{\begin{matrix}[l] & [l^{\perp}] \\ \varphi & \varphi\end{matrix}}{\varphi}\,\mathsf{Exc}_2$$

where $l \in \mathbb{L}$ and φ is any formula, and $l^{\perp}$ denotes the dual of the literal l (so $(c^+)^{\perp} = c^-$ and $(c^-)^{\perp} = c^+$). The rule Exc_1 (*exclusion*) ensures $l^{\perp} \leftrightarrow \neg l$ is derivable; Exc_2 (*case analysis* *on literals*) is the decidability schema for atoms. By a result of Negri and von Plato [9], these two rules together recover full classical logic from the intuitionistic base. The rules are harmonious: Exc_1 is an introduction rule for $\perp$ and Exc_2 functions as a case-elimination rule, and neither extends the expressible consequence relation beyond what the other warrants.

Base-extension Semantics. Bases consist of atomic rules over literals; that is, with $l_i, l \in \mathbb{L}$ and $L_i \subseteq \mathbb{L}$ finite we have $\dfrac{L, L_1 \Rightarrow l_1 \qquad \ldots \qquad L, L_n \Rightarrow l_n}{L \Rightarrow l}$

Derivability in a base $L \vdash_{\mathfrak{B}} l$ is defined inductively over such rules, for any $L \subseteq \mathbb{L}$, together with Exc_1 and Exc_2:

- Exc_1: if $L \vdash_{\mathfrak{B}} l$ and $L \vdash_{\mathfrak{B}} l^{\perp}$, then $L \vdash_{\mathfrak{B}} m$ for every $m \in \mathbb{L}$;
- Exc_2: if $l, L \vdash_{\mathfrak{B}} m$ and $l^{\perp}, L \vdash_{\mathfrak{B}} m$, then $L \vdash_{\mathfrak{B}} m$.

Support is then defined by the clauses of Sandqvist's intuitionistic B-eS verbatim — atomic support reduces to base derivability, conjunction and implication receive their standard proof-theoretic clauses, and disjunction is interpreted as: $\Vdash_{\mathfrak{B}} \phi \vee \psi$ iff for all $\mathfrak{C} \supseteq \mathfrak{B}$ and all $l \in \mathbb{L}$, if $\phi \Vdash_{\mathfrak{C}} l$ and $\psi \Vdash_{\mathfrak{C}} l$ then $\Vdash_{\mathfrak{C}} l$. The support clauses are *identical* to those for intuitionistic logic; the classical character is carried entirely by the base-level duality clauses. Validity is defined by quantification over all bases: $\Gamma \Vdash \phi$ iff $\Gamma \Vdash_{\mathfrak{B}} \phi$ for every base $\mathfrak{B}$.

Soundness and Completeness. The main result of the paper is the following:

Theorem. $\Gamma \Vdash \phi$ *if and only if* $\Gamma \vdash_{\mathsf{NK}^{\pm}} \phi$.

Soundness is proved by induction on $\mathsf{NK}^{\pm}$-derivations; the intuitionistic cases follow directly from Sandqvist [18] and the two classical cases use the duality clauses of base derivability. Completeness adapts Sandqvist's elementary simulation method: a *simulation base* $\mathfrak{N}$ is constructed that encodes the rules of $\mathsf{NK}^{\pm}$ as atomic rules over literals, via a flattening of subformulae to fresh literals. Crucially, since Exc_1 and Exc_2 are built directly into base derivability, they require no separate simulation in $\mathfrak{N}$, and the argument goes through for the full propositional language including $\vee$. The proof is *entirely constructive*, requiring no bar induction, no excluded middle, and no detour through classical model theory; the reasoning appears formalisable in an intuitionistic metatheory, though we leave a precise identification to future work. This meta-level constructivity is independent of the object-level sense in which the semantics is constructive — namely that support is defined without assuming bivalence.

Significance. Several aspects of this work are of direct interest to the proof-theoretic community. First, the bilateral restriction to atoms — rather than uniform bilateralism as in Rumfitt [16] or Restall [15] — is the precise condition that allows Sandqvist's elementary completeness strategy to transfer to classical logic with the *full* propositional language including $\vee$. Prior work either handled only implication and $\perp$ [17], or required technically involved non-constructive proofs [8]. Second, since the support clauses above the atomic level are exactly those of intuitionistic logic, the semantics makes mathematically precise the philosophical thesis that classical logic is intuitionistic logic supplemented by atomic duality. This opens a direct path to systematic study of classical extensions — including classical modal logics — along the lines of the intuitionistic programme developed by Sandqvist, Pym et al. [6, 7], and Eckhardt–Pym [3, 4]. Third, the result addresses Dummett's challenge on entirely anti-realist grounds: the metalogic is constructive, bivalence is not assumed, and the completeness proof witnesses that every classically valid inference is proof-theoretically supported. This semantic approach is complementary to the proof-term and polarity-based traditions for classical logic — the Curry–Howard line of Griffin [10], Parigot [11], and Girard [12], and the polarised calculi developed after them — which pursue constructivity through computation and focusing rather than base-extension meaning.

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