A Survey of Proof-theoretic Semantics[1]

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Prawitz [14] observed that proof theory has two branches. On the one hand, there is the Hilbert-style 'reductive' proof theory that seeks to represents mathematical reasoning using syntactic structures. On the other, there is 'general' proof theory that studies proofs as mathematical objects in their own right. The term 'proof' here need not mean a formal derivation. For example, the BHK interpretation of intuitionism studies proofs without a derivations system.

Proof-theoretic semantics (P-tS) is the area of general proof theory concerned with semantics. It has two projects. First, to provide a semantics of proofs — explaining what makes arguments valid. Second, to give semantics in terms of proofs — defining logical constants via their inferential roles. The term 'proof-theoretic semantics' was popularized by Schroeder-Heister [22] in the early 1990s.

In his foundational work *Investigations into Logical Deduction*, Gentzen [4] remarked that the introduction rules of his natural deduction systems define the meaning of logical constants. This remark is as intuitive as it is enigmatic. Much of P-tS has concerned delivering on this observation. Prawitz [16, 13], following an idea by Lorenzen [9], developed his normalization theory and used it to give a semantics of proofs. Schroeder-Heister [20, 21] has given a summary of this work but we give a general overview of this work here. Let L be a system of proof rules and let $\mathbf{J}$ be a procedure on proof-structures that yields proof-structures (e.g., Prawitz's reductions [16] for normalizing proofs). Given these, we can define validity relative to a 'base' $\mathcal{B}$ and the procedures $\mathbf{J}$ with the following principles:

- There is an *a priori* collection $\mathbf{C}(\mathsf{L})$ of $\langle \mathcal{B}, \mathbf{J} \rangle$-valid proofs of L.
- A completed proof-structure Φ is $\langle \mathcal{B}, \mathbf{J} \rangle$-valid if $\mathbf{J}$ can be applied to Φ to yield an element of $\mathbf{C}(\mathsf{L})$.
- An incomplete proof-structure Φ is $\langle \mathcal{B}, \mathbf{J} \rangle$-valid if any completion of it is $\langle \mathcal{C}, \mathbf{J} \rangle$-valid for any $\mathcal{C} \supseteq \mathcal{B}$.

There are several ways of instantiating these ideas. Prawitz [15] conjectured that when $\mathbf{C}(\mathsf{L})$ is the set of $\mathcal{B}$-derivations together with canonical NJ-derivation and $\mathbf{J}$ is taken to be the reduction operations used in his normalization theorem, this would yield a semantics for intuitionistic proof. This turned out to be false.

One can give a semantics of the logical constants based on Prawitz's semantics of proofs. A formula ϕ is 'supported' by $\mathcal{B}$ iff there is a $\langle \mathcal{B}, \mathbf{J} \rangle$-valid proof Φ concluding ϕ. Piecha et al. [1, 11, 12] studied Prawitz's semantics in this way. For example, disjunction may be expressed as follows:

$$\Vdash_{\mathcal{B}} \phi \vee \psi \qquad \text{iff} \qquad \Vdash_{\mathcal{B}} \phi \text{ or } \Vdash_{\mathcal{B}} \psi$$

It turns out that intuitionistic logic is incomplete for this semantics; that is, Prawitz's conjecture was false. Stafford [23] showed that the result is a semantics for an intermediate logic.

In parallel to this work, Sandqvist [19] developed a similar support relation and called it *base-extension semantics* (B-eS). The biggest difference is the treatment of disjunction for which he uses the following clause instead:

$$\Vdash_{\mathcal{B}} \phi \vee \psi \qquad \text{iff} \qquad \forall \mathcal{C} \supseteq \mathcal{B} \forall P, \text{ if } \phi \Vdash_{\mathcal{C}} P \text{ and } \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P$$

He showed that, depending on the choice of bases $\mathcal{B}$, this is a semantics for intuitionistic logic. Sandqvist [17, 18] has also given analogous semantics for classical logic. The motivation was to give an anti-realist foundation to logic and so in both cases the soundness and completeness are constructively valid.

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This work has been developed tremendously in the recent period. Makinson [10] revisited the B-eS for classical logic and explained its connection to truth-functional account. Eckhardt and Pym [2, 3] used this to develop the B-eS for the standard modal logics. Gheorghiu et al. used the B-eS of intuitionistic logic to give B-eS for substructural logics, first IMLL [7, 8] and then BI [6]. Gheorghiu [5] has also studied the first-order case, following Sandqvist [18]. There are many other results indicating the significance of B-eS for computational logic. These developments suggest that P-tS may offer a robust alternative to model-theoretic accounts, particularly in constructive and computational settings.

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