# Truth, Support, and Arithmetic

ALEXANDER V. GHEORGHIU
*University of Southampton and University College London*
e-mail: a.v.gheorghiu@soton.ac.uk

Gödel's second incompleteness theorem is usually understood as exhibiting a gap between provability and truth [4]. We present a mathematically precise alternative reading in which the gap lies between two notions of consequence internal to a single arithmetical theory.

Working within the proof-theoretic semantics of Sandqvist [6], in which a semantic consequence relation $\Vdash$ is defined compositionally via support in bases of atomic inference rules, we show that this relation and derivability ($\vdash$) diverge for any sufficiently strong, recursively axiomatisable arithmetic theory $A$ in the standard signature $\sigma = \langle 0, S, +, \times \rangle$: one has $A \Vdash \mathrm{Con}(A)$, even though $A \nvdash \mathrm{Con}(A)$. More generally, we establish the uniform reflection principle $A \Vdash \mathrm{Prov}_A(\ulcorner \varphi \urcorner) \to \varphi$ [2].

The proof proceeds by induction on codes of $A$-proofs, analysing the compositional clauses of the support relation. A key new observation is that the finiteness of the signature of arithmetic is what precisely controls the relationship between support and derivability: it prevents the completeness direction of the soundness and completeness theorem [3] from going through, so that semantic support is strictly stronger than derivability. Gödel's theorem is not contradicted; rather, incompleteness arises exactly at the point where completeness breaks down.

We further show that the framework admits an elementary consistency proof for PA [1]: ordinary induction over the natural numbers suffices to construct a consistent base supporting the axioms of PA, yielding PA $\nvdash \bot$ by soundness. This stands in notable contrast to Gentzen's use of transfinite induction up to $\varepsilon_0$.

Together, these results establish that Gödel's incompleteness theorems can be understood as a separation between derivability and proof-theoretic semantic consequence, rather than as a gap between syntactic provability and truth in an independently given model [5].

## References

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