THE PROOF SOCIETY 2026

A Constructive Semantics for Classical Logic

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MOVEMENT I

Background

Completeness and Constructive Metatheory.

THE PROBLEM

Kripke completeness is not constructive

$$\text{Kripke} \vDash \varphi \quad \Longrightarrow \quad \text{IPL} \vdash \varphi$$

Completeness. True — but what does it take to *prove* it?

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Over IZF: Kripke completeness of IPL **implies Markov's Principle**.

MP: $\neg\neg\exists n\, A(n) \to \exists n\, A(n)$ for decidable A — not intuitionistically provable.

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The object logic is intuitionistic; the metatheory that fits the semantics to it is ***not****.*

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Patches: change what counts as a model

VELDMAN 1976

fallible Kripke models

a node may force $\perp$

DE SWART 1976

fallible Beth models

paths, not points

FRIEDMAN 1977

exploding models

the same move, distilled

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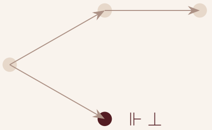

Why it works. The canonical model is built from theories, and constructively one **cannot decide** which are consistent. So keep them all, and let the inconsistent nodes force everything.

Classically: prune the exploding nodes, nothing changes. Constructively, pruning is exactly what is unavailable.

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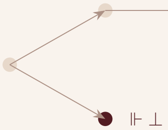

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Classically: prune the exploding nodes, nothing changes. Constructively, pruning is exactly what is unavailable.

*Constructive completeness — for a **weaker semantics**, bent to fit the metatheory.*

THE PROBLEM

Or: start from the other end

Ground the semantics not in *truth* but in **constructions**.

MODEL-THEORETIC

models $\longrightarrow$ truth $\longrightarrow$ $\models$

Meaning is *truth in a model*. Whether the metatheory is constructive is a *target*: checked afterwards — and missed.

Then the semantics is patched until the proof goes through.

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PROOF-THEORETIC

atomic rules $\mathcal{B}$ $\rightarrow$ support $\Vdash_{\mathcal{B}}$ $\rightarrow$ $\Vdash$

Meaning is *justification* — Dummett, Prawitz. Validity is what can be *derived* in atomic inferential practice: a construction, not a truth-value.

Constructivity is the *starting point*: nothing to check afterwards, nothing to patch.

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*This is **proof-theoretic semantics**: no models — only atomic rules, and support.*

SANDQVIST 2015

Base-extension semantics for IPL

BASES

A base $\mathcal{B}$ is a set of atomic rules. Second-level: a rule may *discharge* atoms — P_i finite sets of atoms, p_i, q atoms. First-level: every $P_i = \emptyset$.

A SECOND-LEVEL RULE

$$\frac{\begin{matrix}[P_1] & & [P_n] \\ p_1 & \cdots & p_n\end{matrix}}{q}$$

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$\Vdash_{\mathcal{B}} \bot$	iff $\vdash_{\mathcal{B}} p$ for all p		($\bot$)
$\Vdash_{\mathcal{B}} \varphi \to \psi$	iff $\varphi \Vdash_{\mathcal{B}} \psi$		($\to$)
$\Vdash_{\mathcal{B}} \varphi \wedge \psi$	iff $\Vdash_{\mathcal{B}} \varphi$ and $\Vdash_{\mathcal{B}} \psi$		($\wedge$)
$\Vdash_{\mathcal{B}} \varphi \vee \psi$	iff $\forall \mathcal{C} \supseteq \mathcal{B}, \forall p :\ \varphi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p \Rightarrow \Vdash_{\mathcal{C}} p$		($\vee$)
$\Gamma \Vdash_{\mathcal{B}} \varphi$	iff $\forall \mathcal{C} \supseteq \mathcal{B} :\ \Vdash_{\mathcal{C}} \Gamma \Rightarrow \Vdash_{\mathcal{C}} \varphi$		(Inf)
$\Gamma \Vdash \varphi$	iff $\Gamma \Vdash_{\mathcal{B}} \varphi$ for all $\mathcal{B}$		(Val)

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$\Vdash_{\mathcal{B}} \varphi \to \psi$	iff $\varphi \Vdash_{\mathcal{B}} \psi$		($\to$)
$\Vdash_{\mathcal{B}} \varphi \land \psi$	iff $\Vdash_{\mathcal{B}} \varphi$ and $\Vdash_{\mathcal{B}} \psi$		($\land$)
$\Vdash_{\mathcal{B}} \varphi \lor \psi$	iff $\forall \mathcal{C} \supseteq \mathcal{B}, \forall p :\ \varphi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p \Rightarrow \Vdash_{\mathcal{C}} p$		($\lor$)
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COMPLETENESS

Elementary. Simulate NJ inside a base.

“no canonical models, no König’s lemma, not even bar induction.”

THE QUESTION

From intuitionistic to classical

We have a **constructive** semantics for intuitionistic logic.

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From intuitionistic to classical

We have a **constructive** semantics for intuitionistic logic.

But what constructive semantics for **classical** logic?

MOVEMENT II

Classical Logic

…constructively.

SANDQVIST 2009

Base-extension semantics for CL — without bivalence

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A base $\mathcal{B}$ is a set of atomic rules, now first-level: no discharge.

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\end{array}$$

Bivalence is never assumed — and $\Vdash \, = \, \vdash_{\mathrm{CL}}$.

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$$\frac{p_1 \quad \cdots \quad p_n}{q}$$

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G. 2025: constructive and elementary — includes first-order.

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But what about ***disjunction****?*

STAFFORD ET AL. 2025

Put ∨ back

INSERT SANDQVIST'S ∨-CLAUSE, VERBATIM

$$\vdots$$

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Doubted by de Campos Sanz–Piecha–Schroeder-Heister 2014,
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CURIOUSLY

Same clauses as for IPL; only the bases differ. First-level: CL. Second-level: IPL.

Why? A mystery.

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STAFFORD ET AL. 2025

This gives classical logic! $\Vdash \; = \; \vdash_{\mathrm{CL}}$, with $\wedge, \vee$ primitive.

Only the completeness proof is not constructive.

THE GAP

The map

	WITHOUT $\vee$	WITH $\vee$
FIRST-LEVEL no discharge	CL · constructive Sandqvist 2009 · Gheorghiu 2025	CL · **not** constructive Stafford et al.
SECOND-LEVEL discharge	IPL fragment Sandqvist 2015, restricted	IPL · constructive Sandqvist 2015

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Disjunction wants second-level bases. Classicality, so far, wants first-level ones.

MOVEMENT III

The Fix

Add duality underneath the connectives.

THE FIX

The idea

$$\text{Classical} = \text{Intuitionistic} + \text{Duality}$$

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Don't add classical rules on top of the connectives.

*Add duality **underneath** them — at the atoms.*

THE FIX

Literals

GRAMMAR

$$l \ ::= \ c^+ \mid c^-$$

$c \in \mathcal{C}$ a *content*; c^+ asserts it, c^- denies it. $\mathcal{L}$: the literals.

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$(c^+)^\circ = c$ $\qquad$ $(c^-)^\circ = \neg c$ $\qquad$ $(\varphi * \psi)^\circ = \varphi^\circ * \psi^\circ$

and atoms embed the other way, $p \mapsto p^+$.

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NOT NEGATED ATOMS

c^- is **primitive**: a denial, not $\neg c^+$.

Force lives at the atoms; formulae are logico-grammatical structures of propositions (Wittgenstein 5.4611).

But c^- *behaves* as $\neg c^+$ does — hence "literal".

THE FIX

$\mathrm{NK}^{\pm} \;=\;$ NJ over literals + two rules

Exc_1 · EXCLUSION

$$\frac{l \qquad l^{\perp}}{\perp}$$

a $\perp$-introduction; gives $l^{\perp} \leftrightarrow \neg l$

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Exc_2 · DECIDABILITY

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a case-elimination; $l \vee l^{\perp}$, never stated

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Both **harmonious** — no logical sign out of place. Classicality is carried entirely by the literals.

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THEOREM · $\mathrm{NK}^{\pm}$ IS CLASSICAL LOGIC OVER LITERALS

For all Γ, φ over literals: $\Gamma \vdash_{\mathrm{NK}^{\pm}} \varphi$ iff $\Gamma^{\circ} \vdash_{\mathrm{NK}} \varphi^{\circ}$.

So over atoms, $\vdash_{\mathrm{NK}^{\pm}} = \vdash_{\mathrm{NK}} = \vdash_{\mathrm{CL}}$. Cf. Negri–von Plato: atomic exclusion and cases suffice.

THE FIX

Bases with duality

BASES

A base $\mathcal{B}$ is a set of atomic rules over literals — second-level, exactly as for IPL.

L_i finite sets of literals; l_i, l literals. Written $(L_1 \Rightarrow l_1, \dots, L_n \Rightarrow l_n) \Rightarrow l$.

AN ATOMIC RULE

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DERIVABILITY IN A BASE — THE LEAST RELATION $\vdash_{\mathcal{B}}$ SUCH THAT, FOR ALL $L \subseteq \mathcal{L}$

$$l \in L \quad \Rightarrow \quad L \vdash_{\mathcal{B}} l \qquad \text{(ref)}$$

$$(\Rightarrow l) \in \mathcal{B} \quad \Rightarrow \quad L \vdash_{\mathcal{B}} l \qquad (\text{app}_1)$$

$$(L_1 \Rightarrow l_1, \ldots, L_n \Rightarrow l_n) \Rightarrow l \in \mathcal{B} \text{ and } L, L_i \vdash_{\mathcal{B}} l_i \text{ for all } i \quad \Rightarrow \quad L \vdash_{\mathcal{B}} l \qquad (\text{app}_2)$$

$$L \vdash_{\mathcal{B}} l \text{ and } L \vdash_{\mathcal{B}} l^{\perp} \quad \Rightarrow \quad L \vdash_{\mathcal{B}} m \text{ for all } m \qquad (\text{exc}_1)$$

$$l, L \vdash_{\mathcal{B}} m \text{ and } l^{\perp}, L \vdash_{\mathcal{B}} m \quad \Rightarrow \quad L \vdash_{\mathcal{B}} m \qquad (\text{exc}_2)$$

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Black: Sandqvist 2015, verbatim, over literals. Tinted: Exc_1, Exc_2 in the base — the only change.

THE FIX

Support — unchanged

Bases $\mathcal{B}$: atomic rules over literals $l \in \mathcal{L}$, second-level, with $\mathsf{exc}_1, \mathsf{exc}_2$.

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Support — unchanged

Bases $\mathcal{B}$: atomic rules over literals $l \in \mathcal{L}$, second-level, with $\mathsf{exc}_1, \mathsf{exc}_2$.

$\Vdash_{\mathcal{B}} l$	iff $\vdash_{\mathcal{B}} l$		(At)
$\Vdash_{\mathcal{B}} \bot$	iff $\vdash_{\mathcal{B}} l$ for all l		$(\bot)$
$\Vdash_{\mathcal{B}} \varphi \to \psi$	iff $\varphi \Vdash_{\mathcal{B}} \psi$		$(\to)$
$\Vdash_{\mathcal{B}} \varphi \wedge \psi$	iff $\Vdash_{\mathcal{B}} \varphi$ and $\Vdash_{\mathcal{B}} \psi$		$(\wedge)$
$\Vdash_{\mathcal{B}} \varphi \vee \psi$	iff $\forall \mathcal{C} \supseteq \mathcal{B}, \forall l :\ \varphi \Vdash_{\mathcal{C}} l$ and $\psi \Vdash_{\mathcal{C}} l \Rightarrow \Vdash_{\mathcal{C}} l$		$(\vee)$
$\Gamma \Vdash_{\mathcal{B}} \varphi$	iff $\forall \mathcal{C} \supseteq \mathcal{B} :\ \Vdash_{\mathcal{C}} \Gamma \Rightarrow \Vdash_{\mathcal{C}} \varphi$		(Inf)
$\Gamma \Vdash \varphi$	iff $\Gamma \Vdash_{\mathcal{B}} \varphi$ for all $\mathcal{B}$		(Val)

VERBATIM

Sandqvist's clauses, **letter for letter.**

The classical character lives entirely in $\vdash_{\mathcal{B}}$.

THE FIX

Theorem

$$\Vdash \varphi \quad \Longleftrightarrow \quad \vdash_{\mathrm{NK}^{\pm}} \varphi$$

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$$\Vdash \varphi \quad \Longleftrightarrow \quad \vdash_{\mathrm{NK}^{\pm}} \varphi$$

Completeness, by simulation. Two ingredients:

FLATTENING

Fix φ, with subformulae Ξ. Choose an injection $(-)^{\flat} : \Xi \cup \Xi^{\perp} \cup \{\perp, \top\} \to \mathcal{L}$ with

$$l^{\flat} = l \qquad\qquad (\xi^{\flat})^{\perp} = (\xi^{\perp})^{\flat}$$

Subformulae become literals; literals stay put; duality survives.

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Subformulae become literals; literals stay put; duality survives.

SIMULATION BASE $\mathcal{N}$

Defined inductively on Ξ: one atomic rule for each NJ-rule, stated on the $\xi^{\flat}$.

$\mathsf{Exc}_1, \mathsf{Exc}_2$ need no rule — they already live in $\vdash_{\mathcal{B}}$. That is why $\vee$ goes through.

THE FIX

Completeness — three lemmas

AtComp	$L \Vdash_{\mathcal{B}} l$ iff $L \vdash_{\mathcal{B}} l$	any base $\mathcal{B}$
Flattening	$\Vdash_{\mathcal{N}'} \xi$ iff $\Vdash_{\mathcal{N}'} \xi^{\flat}$	any $\mathcal{N}' \supseteq \mathcal{N}$, $\xi \in \Xi$
Naturalizing	$\vdash_{\mathcal{N}} \xi^{\flat} \Rightarrow \vdash_{\mathrm{NK}^{\pm}} \xi$	$\mathcal{N}$ proves nothing $\mathrm{NK}^{\pm}$ cannot

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ASSEMBLY

$$\Vdash \varphi \quad \Rightarrow \quad \Vdash_{\mathcal{N}} \varphi \quad \xRightarrow{\text{Flat}} \quad \Vdash_{\mathcal{N}} \varphi^\flat \quad \xRightarrow{\text{AtComp}} \quad \vdash_{\mathcal{N}} \varphi^\flat \quad \xRightarrow{\text{Nat}} \quad \vdash_{\mathrm{NK}^\pm} \varphi$$

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no bar induction. no models. no excluded middle.

MOVEMENT IV

Closing

What it buys, and what remains.

CLOSING

The map, completed

	WITHOUT ∨	WITH ∨
FIRST-LEVEL	CL · constructive Sandqvist 2009 · Gheorghiu 2025	CL · not constructive Stafford et al.
SECOND-LEVEL	IPL fragment	IPL · constructive Sandqvist 2015
SECOND-LEVEL over literals	CL · **constructive** · **elementary** · full language this talk	

CLOSING

Why it matters

- We get **full classical logic** — $\wedge, \vee, \to, \bot$ primitive — with a **constructive**, elementary completeness proof.

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- Dummett's challenge, met on **anti-realist** grounds — meaning *and* metatheory.

COMPARE

Griffin 1990, Parigot 1992, Girard 1991 made classical logic constructive through its *proofs*: control operators typed by Peirce's law, the $\lambda\mu$-calculus, a polarised decomposition via linear logic. Here the constructive content lies in the **meaning**, not the proofs. Same aim, different route.

CLOSING

Open

FORMALISE

Pin down the metatheory.

The proof uses no bar induction, no excluded middle, no models — but which fragment of HA or IZF actually proves completeness?

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Support vs. fallible models.

Veldman's exploding nodes also let completeness go through constructively. Is $\Vdash_{\mathcal{B}}$ a fallible model in disguise — or is the relationship between B-eS and Kripke semantics something else entirely?

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Veldman's exploding nodes also let completeness go through constructively. Is $\Vdash_{\mathcal{B}}$ a fallible model in disguise — or is the relationship between B-eS and Kripke semantics something else entirely?

Which brings us back to the first slide.

REFERENCES & THANKS

Sources

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Thank you.

BACKUPS

For questions

Peirce, disjunction, involution, Markov.

BACKUP

Peirce's law in $\mathrm{NK}^{\pm}$ (atoms l, m)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{[l]^3 \qquad [l^{\perp}]^2}{\perp}\,\mathsf{Exc}_1
      }{m}\,\perp\mathrm{E}
    }{l \to m}\,\to\mathrm{I}^3
    \qquad
    [(l \to m) \to l]^1
  }{l}\,\to\mathrm{E}
  \qquad [l]^2
}{
  \dfrac{l}{((l \to m) \to l) \to l}\,\to\mathrm{I}^1
}\,\mathsf{Exc}_2^2
$$

Case analysis on l at the leaves; no reductio anywhere.

BACKUP

Why $\vee$ resists the $\to/\bot$ semantics

- Sandqvist's $\vee$-clause quantifies over *extensions* and *atoms*: it is $\vee$-*elimination* made semantic.
- Completeness by simulation needs $\vee$E as an atomic rule — i.e. a rule *with discharge*: a second-level base.
- But over plain atoms, second-level bases make the $\to/\bot$ fragment intuitionistic (Peirce fails).
- Over literals with $\mathsf{exc}_1, \mathsf{exc}_2$, discharge is harmless: classicality is already in $\vdash_{\mathcal{B}}$.

Disjunction and classicality stop competing for the same base.

BACKUP

Duality is not an involution on syntax

$$((l \to m)^{\perp})^{\perp} = (l \wedge m^{\perp})^{\perp} =$$
$$l^{\perp} \vee m \neq l \to m$$

- It *is* an involution on truth value: $v(\varphi^{\perp}) = -v(\varphi)$.
- The arXiv version handles this with $\varphi \cong \psi$ iff $\varphi^{\perp} = \psi^{\perp}$ in the naturalising lemma.
- With $\mathsf{Exc}_1, \mathsf{Exc}_2$ on literals only, no duality on *formulae* is needed at all.

BACKUP

McCarty 1994, precisely

- Over IZF: strong completeness of IPL (indeed of the $\to/\neg$ fragment) for Kripke semantics implies Markov's Principle.
- Weak completeness for Kripke semantics is also refutable under Church's Thesis.
- Kreisel 1962: completeness of intuitionistic predicate logic for validity implies MP.

None of this touches $\Vdash$*: there is no valuation to be decided.*