SLSS 2026

Towards a Proof-theoretic Foundation for Mathematics

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THE CONTEXT

Where I come from

I came to logic through the **foundations of mathematics**. The early twentieth century offered three great answers:

LOGICISM

Frege · Russell

Mathematics is logic in disguise: numbers are logical objects, arithmetic pure reason.

FORMALISM

Hilbert

Mathematics is a calculus of signs, to be secured by finitary consistency proofs.

INTUITIONISM

Brouwer · Heyting

Mathematics is mental construction; logic answers to it, not the other way round.

All three are problematic — I shan't relitigate how; but the *ambition* was right.

Recent work in logic makes a fourth answer possible: ***inferentialism****.*

THE TEST CASE

Gödel's incompleteness theorems

Let A be a consistent, effectively axiomatised theory with enough arithmetic (e.g. Q, PA).

FIRST THEOREM (1931)

There is a sentence φ with

$$A \nvdash \varphi \quad \text{and} \quad A \nvdash \neg\varphi$$

Some arithmetical sentences the theory can neither prove nor refute.

SECOND THEOREM (1931)

One such sentence can be named:

$$A \nvdash \mathrm{Con}(A)$$

No such theory proves its own consistency.

The received moral: arithmetical ***truth*** *outruns* ***proof****. — But truth where?*

PROLOGUE

Meaning as Use

What if signs mean what we do with them?

PROLOGUE

Where does the meaning of a sign live?

Two pictures of what a sign — $\to$, $\forall$, or indeed 0, S, $+$ — *means*:

THE REFERENTIAL PICTURE

A sign means what it *stands for.*

Frege's *Bedeutung*, Tarski's model theory: signs denote objects and functions in a structure; truth is correspondence. Meaning first, use after.

THE INFERENTIALIST PICTURE

A sign means how it is **used** — in *deduction.*

Wittgenstein: meaning is *use.* Here in the narrow, deductive sense — the *rules of inference* governing the sign — not Brandom's broad social-normative inferentialism. Use first, 'truth' after.

This talk adopts the second picture — and takes it mathematically seriously.

PROLOGUE

An anti-realist view — so, intuitionism?

Meaning as **use** is an anti-realist view: no appeal to a ready-made world of referents, no bivalence assumed. Historically, it lends itself most naturally to **intuitionism** — giving up bivalence has seemed to mean giving up classical logic.

> "In the resolution of the conflict between [the view that generally accepted classical modes of inference ought to be theoretically accommodated, and the demand that any such accommodation be achieved without recourse to bivalence] lies, as I see it, one of the most fundamental and intractable problems in the theory of meaning; indeed, in all philosophy."
>
> — Michael Dummett, address at the British Academy, 1973

PROLOGUE

For the logical signs, this is a finished story

Gentzen · Prawitz · Dummett

decades of work towards
proof-theoretic semantics (P-tS)

realized by →

Sandqvist

base-extension semantics
— the culmination

The meaning of $\to$, $\wedge$, $\forall$, ... is fixed by **use** — the rules of inference —
not by reference to a model. And the semantics it yields is *classical*, without bivalence.

Decades of philosophy, now theorems.

PROLOGUE

So — what is the meaning of *arithmetic*?

- If meaning is deductive use, then 0, S, $+$, $\times$ mean the atomic rules by which we use them.
- Two consequence relations then demand attention:

$\vdash$ derivability from axioms $\Vdash$ support by meaning

- Arithmetical *truth* is then something to be **explained** — not presupposed as 'what holds in $\mathbb{N}$'.

THE SIGNS, IN USE

$$\frac{}{x + 0 = x} \qquad \frac{}{x + Sy = S(x{+}y)}$$

$$\frac{Sx = Sy}{x = y}$$

the meaning of $+$ and S: their defining equations, read as rules of use

The puzzle, in a sentence: can arithmetical truth be grounded in inference rather than reference?

THE TEST CASE

Gödel II, re-read

The crucible for any account of arithmetical meaning: Gödel's second incompleteness theorem.

THE STANDARD READING

Gödel II is a gap between *provability* and *truth*.

A theory cannot prove its own consistency; truth lives *outside*, in a model of $\mathbb{N}$.

THIS TALK

For an inferentialist this is confused... *isn't arithmetic 'truth' what we are trying to explain?*

Thesis: there are *two* proof-theoretic consequence relations to consider

— Derivability $\vdash$ and support $\Vdash$.

THE ITINERARY

Three questions

If we take the inferentialist picture seriously, three questions force themselves on us:

QUESTION 1

Does a semantics of *rules* go beyond the rules?

Surely it just re-describes derivability…

QUESTION 2

What, then, *is* Gödel's gap?

If truth is not external — a gap between *what* and *what*?

QUESTION 3

Can *meaning alone* prove consistency?

Gentzen paid in ordinals. What is the price here?

*One thread will answer all three: watching a completeness proof **fail**.*

MOVEMENT I

The Machinery

The support relation, formally.

THE MACHINERY

One new ingredient: support $\Vdash$

- A base $\mathcal{B}$ is a *finite* set of atomic inference rules
  — a codification of the *use* of the atomic signs.
- Support $\Vdash$ is defined inductively by clauses for the connectives.
- It is proof-theoretic: no model, no interpretations.

A BASE $\mathcal{B}$

$$\frac{}{\mathsf{N}(0)} \qquad \frac{\mathsf{N}(x)}{\mathsf{N}(Sx)}$$

$$\frac{\mathsf{N}(x) \quad \mathsf{N}(y)}{\mathsf{N}(x+y)}$$

atomic rules over $\langle 0, S, +, \times \rangle$

THE MACHINERY

Support, concretely

Relative to a base $\mathcal{B}$, with base-extensions $\mathcal{C} \supseteq \mathcal{B}$:

$$\begin{array}{ll}
\Vdash_{\mathcal{B}} p & \text{iff } \vdash_{\mathcal{B}} p \\
\Vdash_{\mathcal{B}} \varphi \to \psi & \text{iff } \forall \mathcal{C} \supseteq \mathcal{B} : \Vdash_{\mathcal{C}} \varphi \Rightarrow \Vdash_{\mathcal{C}} \psi \\
\Vdash_{\mathcal{B}} \forall x\, \varphi & \text{iff } \Vdash_{\mathcal{B}} \varphi[t] \;\text{ for all closed } t \\
\Vdash_{\mathcal{B}} \bot & \text{iff } \Vdash_{\mathcal{B}} p \;\text{ for all atoms } p \\
\Gamma \Vdash \varphi & \text{iff } \Gamma \Vdash_{\mathcal{B}} \varphi \;\text{ for all bases } \mathcal{B}
\end{array}$$

LOOKS FAMILIAR?

Yes, this resembles *Kripke semantics* — but it really is **not**.

Support validates the **law of excluded middle**: $\varphi \vee \neg\varphi$.

QUESTION 1, ANSWERED?

Support is... just derivability

$$\Gamma \Vdash \varphi \quad \Longleftrightarrow \quad \Gamma \vdash \varphi \qquad \text{(FOL)}$$

- Soundness *and* completeness for *classical* first-order logic — constructively (Gheorghiu, *Logic Journal of the IGPL* **33**, 2025).
- So the deflationary suspicion looks *right*: support merely re-presents derivability. Nothing gained; the talk should end here.
- Except — the completeness proof only goes through under a **condition**.
  Let me show you the one piece of the proof that matters. Watch where it creaks.

ONE PROOF, SLOWLY

Completeness must simulate *freshness*

- Completeness: from $\Gamma \Vdash \varphi$, *build* a derivation. The strategy: make the base simulate the proof system.
- Every rule simulates smoothly — except $\forall$-introduction, which turns on an eigenvariable: a name about which *nothing* is assumed.
- The semantics can manufacture one: **extend the signature** by a new constant c — no rule in any base mentions it, so c is 'arbitrary'.

THE CRUX

$$\frac{\varphi[y/x]}{\forall x\, \varphi}\; y \text{ fresh}$$

vs.

$$\Vdash_{\mathcal{B}} \forall x\, \varphi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi[t]\,, \text{ all closed } t$$

a rule about *fresh names*, mirrored by a clause about *all names*

The whole proof hangs on an indefinite supply of ***fresh*** *closed terms.*

THE TWIST

Arithmetic has no fresh names

- In $\langle 0, S, +, \times \rangle$, *every* closed term already names a number.
- Nothing is left to be 'arbitrary': the simulation **cannot be run**.
- Completeness fails — support comes apart from derivability:

$$\vdash \; \subsetneq \; \Vdash$$

$\cancel{c_1,\ c_2,\ c_3,\ \ldots}$

no fresh symbols available

$0,\ S0,\ SS0,\ SSS0,\ \ldots$

every term already names a number

Not a defect — the ***discovery*** *of the talk.*

THE TWIST

“So just add fresh constants!”

A fair objection — let’s try it. Enrich the signature with $c_1, c_2, c_3, \ldots$:

ESCAPE ROUTE 1

Leave the c_i loose.

Completeness returns — but now consistent bases support *equations the theory does not prove.*

Equational completeness breaks.

ESCAPE ROUTE 2

Pin them down: add $c_i = c_0$.

Equations behave — but every constant is now *fixed by the theory*: the fresh reserve is gone.

Completeness fails again.

The divergence is not an artefact of notation — it is ***intrinsic*** *to arithmetic.*

MOVEMENT II

The Payoffs

What we learn, and what we can now do.

PAYOFF A · INCOMPLETENESS

Support strictly exceeds derivability

In the standard signature $\langle 0, S, +, \times \rangle$, the completeness condition **fails** for any **sufficiently strong** theory — like Q or PA — which implicitly uses all closed terms. Hence:

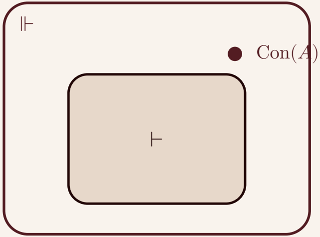
$$A \Vdash \mathrm{Con}(A)$$

... but $A \nvdash \mathrm{Con}(A)$.

And more generally, the uniform reflection principle:

$$A \Vdash (\mathrm{Prov}_A(\ulcorner \varphi \urcorner) \to \varphi)$$

Unprovable truths, **supported** from within.



$\mathrm{Con}(A)$ sits in the gap: supported, not derived.

- **Sufficiently strong** A: $\vdash_A \overline{m} = \overline{n}$ iff $m = n$; and every Δ_0-sentence is decided ($\vdash_A \varphi$ or $\vdash_A \neg\varphi$). e.g. Q, PA.

PAYOFF A · READING

Reframing Gödel

THE OLD PICTURE

Theory $\vdash \cdots$

$\cdots \not\to \cdots$

Truth in $\mathbb{N}$ (external)

Truth lives outside, in a model.

THE NEW PICTURE

one theory

$$\xrightarrow{\quad\vdash\quad}$$

$$\xrightarrow{\quad\Vdash\quad}$$

Two internal arrows: a **semantics–proof** gap, not a truth–theory gap.

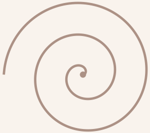
Gödel is not contradicted: the gap appears ***exactly where completeness breaks.***

Dummett's 'indefinite extensibility' of number, made formal: each Gödelian step unfolds meaning already in the base.

PAYOFF B · IN CONTEXT

Consistency without the ordinals

GENTZEN, 1936



transfinite induction to ε_0

HERE

ordinary induction on $\mathbb{N}$

no ε_0.

Wright's dilemma — question-begging or non-suasive? — sits on *neither* horn: the proof unpacks meanings internal to arithmetic.

PAYOFF B · WHAT'S NEW

An elementary consistency proof for PA

- PA is axiomatised by Horn clauses, so we can construct a base $\mathcal{A}$ with

$$\Vdash_{\mathcal{A}} \varphi \quad \text{iff} \quad \mathrm{PA} \Vdash \varphi.$$

- $\mathcal{A}$ is simply structured, one checks by *induction* that it is consistent: $\nVdash_{\mathcal{A}} \bot$.
- Therefore, $\mathrm{PA} \nVdash \bot$.
- *While completeness fails in the standard signature, soundness still holds:*

$$\mathrm{PA} \vdash \varphi \qquad \text{implies} \qquad \mathrm{PA} \Vdash \varphi$$

.

- Contraposition at $\varphi = \bot$ gives $\mathrm{PA} \nvdash \bot$. **(consistency)**

MOVEMENT III

The Reckoning

What it's worth — and what it costs.

THE RECKONING

So — what *is* the meaning of arithmetic?

- The arithmetical signs mean their rules of use — the base $\mathcal{A}$: the axioms of PA, read as atomic rules.
- Arithmetical *consequence* is support relative to that meaning: $\mathrm{PA} \Vdash \varphi$.
- Arithmetical *truth* is support by the meaning alone: $\Vdash_{\mathcal{A}} \varphi$.
  No model of $\mathbb{N}$ required — and no bivalence assumed.
- Derivability $\vdash$ is its *incomplete* shadow — Gödel II marks exactly where the shadow falls short.

Arithmetic means what we do with it — and that meaning outruns any axiomatisation of it.

THE RECKONING

What was known, and what is new

WHAT WAS KNOWN

The *semantics itself*: base-extension semantics, and classical logic without bivalence.

Sandqvist — crowning the Gentzen–Prawitz–Dummett programme.

WHAT I SHOW

Everything the semantics is made to **do** here:

constructive completeness for classical FOL; the diagnosis of exactly where it fails; $A \Vdash \mathrm{Con}(A)$ — incompleteness relocated; the elementary consistency of PA.

The one new idea: watch what completeness needs — fresh names — and notice arithmetic cannot supply them.

THE RECKONING

Compare the bills

All of this from one relation, $\Vdash$ — and at what price? The usual routes bill far more:

INCOMPLETENESS

Accept that **truth** outruns **proof** in mathematics... without further explanation of why?

CONSISTENCY

Show that PA is consistent by transfinite induction.

Accept large infinities

SUPPORT

Adopt one principle — meaning as **use**.

Proof-theoretic Semantics.

Incompleteness and consistency become simple consequences of the setup.

REFERENCES & THANKS

Sources

- *Classical Arithmetic without Bivalence.* arXiv:2506.22326.
- *On the Concept of Arithmetic Consequence.* arXiv:2603.09900.
- *Proof-theoretic Semantics for First-order Logic.* Logic Journal of the IGPL **33** (2025).
- Sandqvist, *Classical logic without bivalence.* Analysis **69** (2009).

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Thank you.

THE HORIZON

Three things I cannot yet do

- **Other undecidable sentences.** Are Gödel sentences, and instances of *uniform* reflection, supported in this sense? The machinery is all here — pick a sentence and run it. Genuinely tractable.
- **Iterated reflection & ordinal analysis.** Can the Kreisel–Feferman progressions be recast as the *unfolding of meaning already in the base* — not new commitments? If you work on reflection principles or ordinal analysis, I suspect a bridge, and I don't yet see it.
- **Second-order fragments.** Extending support to fragments of second-order logic — exactly where Väänänen's worries about the set-theoretic presuppositions of full SOL become live.

If any of these catches you — find me at coffee.