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Truth, Support, and Arithmetic

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THE PICTURE

The standard reading, and the claim

I want to review Gödel II.

THE STANDARD READING

Gödel II is a gap between *provability* and *truth*.

A theory cannot prove its own consistency; truth lives *outside*, in a model of $\mathbb{N}$.

THIS TALK

This seems confused... *isn't arithemtic 'truth' what we are trying to explain?*

Thesis: there are *two* proof-theoretic consequence relations to consider

— Derivability $\vdash$ and support $\Vdash$.

ROADMAP

Three results, one new tool

What does ‘support’ get us?

COMPLETENESS

$\Vdash = \vdash$ (FOL)

Support *is* derivability, under certain conditions.

INCOMPLETENESS

$\Vdash \supsetneq \vdash$

$A \Vdash \mathrm{Con}(A)$, yet $A \nvdash \mathrm{Con}(A)$.

CONSISTENCY

$\mathrm{PA} \nvdash \bot$ (PA)

Elementary; *no* ε_0-induction.

- A precise analysis of **where** completeness fails — exactly what licenses the incompleteness.
- The incompleteness of a *theory* is no different from a *logic* incomplete for a semantics — not a separate mystery.

MOVEMENT I

The Machinery

The support relation, formally.

THE MACHINERY

An old programme, realized

Gentzen · Prawitz · Dummett
the programme of
proof-theoretic semantics

$\xrightarrow{\text{realizes}}$

Sandqvist
base-extension semantics

Meaning is fixed by **use** — the rules of inference — not by reference to a model.

We defer the philosophy; here the payoff is ***mathematical.***

THE MACHINERY

One new ingredient: support $\Vdash$

- A base $\mathcal{B}$ is a *finite* set of atomic inference rules.
- Support $\Vdash$ is defined inductively by clauses for the connectives.
- It is proof-theoretic: no model, no interpretations.

A BASE $\mathcal{B}$

$$\frac{}{\mathsf{N}(0)} \qquad \frac{\mathsf{N}(x)}{\mathsf{N}(Sx)}$$

$$\frac{\mathsf{N}(x) \quad \mathsf{N}(y)}{\mathsf{N}(x+y)}$$

atomic rules over $\langle 0, S, +, \times \rangle$

THE MACHINERY

Support, concretely

Relative to a base $\mathcal{B}$, with base-extensions $\mathcal{C} \supseteq \mathcal{B}$:

$\Vdash_{\mathcal{B}} p$	iff $\vdash_{\mathcal{B}} p$
$\Vdash_{\mathcal{B}} \varphi \to \psi$	iff $\forall\, \mathcal{C} \supseteq \mathcal{B} : \Vdash_{\mathcal{C}} \varphi \Rightarrow \Vdash_{\mathcal{C}} \psi$
$\Vdash_{\mathcal{B}} \forall x\, \varphi$	iff $\Vdash_{\mathcal{B}} \varphi[t]$ for all closed t
$\Vdash_{\mathcal{B}} \bot$	iff $\Vdash_{\mathcal{B}} p$ for all atoms p
$\Gamma \Vdash \varphi$	iff $\Gamma \Vdash_{\mathcal{B}} \varphi$ for all bases $\mathcal{B}$

LOOKS FAMILIAR?

Yes, this resembles *Kripke semantics* — but it really is **not**.

Support validates the **law of excluded middle**: $\varphi \vee \neg\varphi$.

THE MACHINERY

For first-order logic, support is derivability

$$\Gamma \Vdash \varphi \iff \Gamma \vdash \varphi \qquad \text{(FOL)}$$

- Soundness *and* completeness for *classical* first-order logic (Gheorghiu, *Logic Journal of the IGPL* **33**, 2025).
- This proof is *constructive*!
- So far, support is just a *re-presentation* of derivability — nothing gained.
- *…but completeness has a condition.*

THE TWIST

Size of the signature

- Completeness proceeds by *simulating* proofs.
- How do we simulate the use of *eigenvariables*?
- Completeness holds *only if* there is an indefinite supply of fresh closed terms; otherwise,

$$\vdash \; \subsetneq \; \Vdash$$

$\cancel{c_1,\ c_2,\ c_3,\ \ldots}$

no fresh symbols available

$$0,\ S0,\ SS0,\ SSS0,\ \ldots$$

every term already names a number

MOVEMENT II

The Payoffs

What we learn, and what we can now do.

PAYOFF A · INCOMPLETENESS

Support strictly exceeds derivability

In the standard signature $\langle 0, S, +, \times \rangle$, the completeness condition **fails** for any **sufficiently strong** theory — like Q or PA — which implicitly uses all closed terms. Hence:

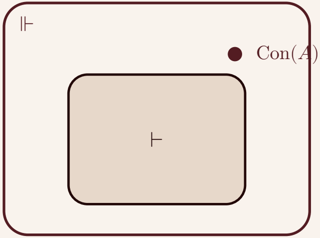
$$A \Vdash \mathrm{Con}(A)$$

... but $A \nvdash \mathrm{Con}(A)$.

And more generally, the uniform reflection principle:

$$A \Vdash (\mathrm{Prov}_A(\ulcorner \varphi \urcorner) \to \varphi)$$

Unprovable truths, **supported** from within.



Con(A) sits in the gap: supported, not derived.

▶ **Sufficiently strong** A: $\vdash_A \overline{m} = \overline{n}$ iff $m = n$; and every Δ_0-sentence is decided ($\vdash_A \varphi$ or $\vdash_A \neg\varphi$). e.g. Q, PA.

PAYOFF A · READING

Reframing Gödel

THE OLD PICTURE

Theory $\vdash$ $\cdots$

$\cdots \quad \nrightarrow \quad \cdots$

Truth in $\mathbb{N}$ (external)

Truth lives outside, in a model.

THE NEW PICTURE

one theory

$$\xrightarrow{\quad\quad\vdash\quad\quad}$$

$$\xrightarrow{\quad\quad\Vdash\quad\quad}$$

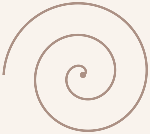
Two internal arrows; the gap is **within**.

Gödel is not contradicted: the gap appears ***exactly where completeness breaks.***

PAYOFF B · IN CONTEXT

Consistency without the ordinals

GENTZEN, 1936



transfinite induction to ε_0

HERE

ordinary induction on $\mathbb{N}$

no ε_0.

PAYOFF B · WHAT'S NEW

An elementary consistency proof for PA

- PA is axiomatised by Horn clauses, so we can construct a base $\mathcal{A}$ with

$$\Vdash_{\mathcal{A}} \varphi \quad \text{iff} \quad \mathrm{PA} \Vdash \varphi.$$

- $\mathcal{A}$ is simply structured, one checks by *induction* that it is consistent: $\nVdash_{\mathcal{A}} \bot$.
- Therefore, $\mathrm{PA} \nVdash \bot$.
- *While completeness fails in the standard signature, soundness still holds:*

$$\mathrm{PA} \vdash \varphi \qquad \text{implies} \qquad \mathrm{PA} \Vdash \varphi$$

.

- Contraposition at $\varphi = \bot$ gives $\mathrm{PA} \nvdash \bot$. **(consistency)**

MOVEMENT III

The Reckoning

What it's worth — and what it costs.

THE RECKONING

The bargain

All of this — from *one* new relation, $\Vdash$:

THE PAYOFFS

- A precise condition for when support *is* derivability
- Explaining the Gödel's gap proof-theoretically
- An elementary consistency proof for PA.

THE ONLY COST

Adopt **proof-theoretic semantics** — meaning as *use*, not reference.

No model theory; no ordinal analysis.

The price of a new way to study mathematics — if I may be grandiose.

THE RECKONING

Compare the bills

To explain the same phenomena, the usual routes ask for far more:

INCOMPLETENESS

Accept that **truth** outruns **proof** in mathematics... without futher explanation of why?

CONSISTENCY

Show that PA is consistent by transfinite induction.

Accept large infinities

SUPPORT

Adopt one principle — meaning as **use**.

Proof-theoretic Semantics.

Incompleteness and consistency become simple consequences of the setup.

REFERENCES & THANKS

Sources

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Thank you.