PART 2: AUTOMATED REASONING · CHAPTER 6

Model Theory

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CHAPTER 6

Model Theory

True here, true everywhere, and the bridge.

OPENING

Tuesday: a claim everyone believes

CASE: THE REQUEST

You look after the platform at a company of five, and you want never to be surprised by your own system. On Tuesday the finance director asks you to confirm something everyone takes for granted: *everybody earns less than their boss.*

EMPLOYEE	BOSS	ENGINEER?	MANAGER?	SALARY
Thorne	Thorne	no	yes	200
Finch	Thorne	no	yes	120
Pratt	Thorne	no	no	95
Bell	Finch	yes	no	90
Cole	Finch	yes	no	85

Figure 6.1: the personnel table on Tuesday.

OPENING

Ask for the rows that break it

```
SELECT e.name, b.name AS boss
FROM   staff e JOIN staff b ON e.boss = b.name
WHERE  NOT (b.salary > e.salary);
```

- One row comes back: `Thorne | Thorne`. The chief executive is her own boss, and she does not earn more than herself.
- Add `AND e.name <> 'Thorne'`; the query returns nothing; the claim holds.

The finance director answers with one line: “Can that ever stop being true?”

OPENING

Two ideas, kept apart from the start

TRUE HERE

True of *this* table, on *this* Tuesday. A matter of looking: the query engine settles it in a fraction of a second.

TRUE EVERYWHERE THE CONSTRAINTS ALLOW

True of *every* table the system can produce, given its HR policy. Infinitely many tables, most of which nobody will see; no amount of looking settles it.

The table changes continually. He does not want to know whether the claim is true now.

OPENING

Three questions

1. What does it mean for a sentence to be **true of a system**?
2. What does it mean for a sentence to **follow from the requirements** on a system, so that it is true of every system that meets them?
3. Is there a **finite way to show** that it does?

Two definitions and a theorem, though the first rules of proof we try fall short.

OPENING

Where we are going

SECTION	WHAT IT DOES
6.1	Syntax: a language in which sentences about a system are written.
6.2	Structures: a system as a mathematical object.
6.3	Satisfaction: when a sentence is true in a structure.
6.4	Entailment: true in every structure in which a theory is true.
6.5	Soundness of NJ: derivations are safe.
6.6	The law of the excluded middle: NJ is not enough; the missing law, and NK.
6.7	Completeness of NK: a sentence follows exactly when it has a derivation.
6.8	Classical and intuitionistic logic: what the extra law costs.

SECTION 6.1

Syntax

A vocabulary that every table can read.

6.1 · SYNTAX

A claim about every table

- The rules of the personnel system are meant to hold of every table it can produce, of five people or five hundred, and each table must be able to read them.
- So the claim cannot be written in terms of Thorne and Finch. It needs a vocabulary that any table of the right shape can interpret in its own way.
- A name for the chief executive, whoever she is; a symbol for *the boss of*; symbols for *is an engineer* and *earns more than*.

Such a vocabulary is a signature. *With it we name things, say things about them, and combine what we have said.*

6.1 · SYNTAX

Definition 6.1: signature

A signature is the vocabulary that sentences about one kind of system may use: names for particular things, symbols for functions and symbols for relations, each with its number of places.

A **signature** is a quadruple $\Sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ar})$ in which

- $\mathcal{C}$, $\mathcal{F}$, $\mathcal{R}$ are pairwise disjoint sets of **constant**, **function** and **relation symbols**;
- $\mathrm{ar} : \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ is the **arity**, with $\mathrm{ar}(f) \geq 1$ for every $f \in \mathcal{F}$.

A relation symbol of arity 0 is a **propositional letter**. Σ is **countable** if $\mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$ is.

Given only the symbol f, nobody could tell whether it takes one input, two, or a million.

6.1 · SYNTAX

Names first

- Fix a signature $\Sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ar})$ and a countably infinite set Var of variables $x, y, z, x_1, x_2, \ldots$, disjoint from the symbols of Σ.
- Constants and variables name elements of the domain.
- Function symbols build new names from old ones.

6.1 · SYNTAX

Definition 6.2: terms

The Σ-**terms** are given by the grammar

$$t \ ::= \ x \ \mid \ c \ \mid \ f(t_1, \ldots, t_n)$$

where x is a variable, c a constant symbol of Σ, and f a function symbol of Σ with $\mathrm{ar}(f) = n$. We write Term_Σ for the set of Σ-terms.

6.1 · SYNTAX

Saying something

- A relation applied to terms is the smallest statement, an atomic formula: $\mathsf{Richer}(\mathsf{boss}(x), x)$ says that the boss of x earns more than x.
- $\bot$ says something false outright.
- The logical symbols combine atomic formulas into complex ones.

Letters: Greek φ, ψ, χ for formulas; Latin p, q for propositional letters.

6.1 · SYNTAX

Definition 6.3: formulas

The Σ-**formulas** are given by the grammar

$$\varphi, \psi \ ::= \ \bot \mid R(t_1, \dots, t_n) \mid (\varphi \to \psi) \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid \forall x\, \varphi \mid \exists x\, \varphi$$

where $\mathrm{ar}(R) = n$, the t_i are Σ-terms and x is a variable; Form_Σ is the set of them. $\bot$ and $R(t_1, \dots, t_n)$ are **atomic**; for a propositional letter, $p()$ is written p.

ABBREVIATION	STANDS FOR
$\neg\varphi$	$\varphi \to \bot$
$\varphi \leftrightarrow \psi$	$(\varphi \to \psi) \wedge (\psi \to \varphi)$
$t_1 \neq t_2$	$\neg(t_1 = t_2)$, when Σ contains $=$

6.1 · SYNTAX

Keeping brackets down

- Familiar two-place symbols go between their arguments: $x + y < z$ is $<(+(x, y), z)$, and $t_1 = t_2$ is $=(t_1, t_2)$.
- The outermost pair of brackets is dropped.
- $\wedge$ and $\vee$ take precedence over $\rightarrow$, which groups to the right; $\neg$, $\forall$ and $\exists$ take precedence over everything else.

A symbol that takes precedence applies first, to as little as it can, as $\times$ *does in* $2 + 3 \times 4$.

6.1 · SYNTAX

Example 6.4: the company signature

CASE: THE COMPANY

Harriet Ashworth looks after the platform at the company of the opening, and the personnel table is hers.

$$\Sigma_{\mathrm{C}} = \Big(\{\mathsf{ceo}\},\ \{\mathsf{boss}\},\ \{=, \mathsf{Eng}, \mathsf{Mgr}, \mathsf{Richer}\},\ \mathrm{ar}\Big)$$

- $\mathrm{ar}(\mathsf{boss}) = \mathrm{ar}(\mathsf{Eng}) = \mathrm{ar}(\mathsf{Mgr}) = 1$ and $\mathrm{ar}(=) = \mathrm{ar}(\mathsf{Richer}) = 2$. $\mathsf{Richer}(x, y)$ reads "x earns more than y".
- $=$ is a two-place relation symbol **like the others**, written between its arguments as usual.

6.1 · SYNTAX

Example 6.4: terms

- ceo, $\mathsf{boss}(\mathsf{ceo})$ and $\mathsf{boss}(\mathsf{boss}(\mathsf{ceo}))$ name the chief executive, her boss and her boss's boss, whoever those turn out to be; $\mathsf{boss}(x)$ names the boss of whoever x is.
- $\mathsf{boss}(\mathsf{ceo}, \mathsf{ceo})$ is not a term, since $\mathrm{ar}(\mathsf{boss}) = 1$. Nor is $\mathsf{Eng}(\mathsf{ceo})$: it says something rather than naming something.
- $\mathsf{Eng}(\mathsf{ceo})$ is an atomic formula, "the chief executive is an engineer", and so is $\mathsf{Richer}(\mathsf{boss}(x), x)$, "the boss of x earns more than x".

6.1 · SYNTAX

Example 6.4: four sentences in English

FORMULA	ENGLISH
$\forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$	everyone earns less than their boss
$\forall x\, (x \neq \mathsf{ceo} \to \mathsf{Richer}(\mathsf{boss}(x), x))$	everyone except the chief executive earns less than their boss
$\forall x\, (\mathsf{Eng}(x) \to \mathsf{Mgr}(\mathsf{boss}(x)))$	every engineer reports to a manager
$\exists x\, (\mathsf{Mgr}(x) \wedge \mathsf{boss}(x) \neq x)$	some manager has a boss other than themselves

Figure 6.2: four sentences and their English.

6.1 · SYNTAX

Example 6.5: the counting numbers

$$\Sigma_{\mathrm{A}} = \Big(\{0\},\ \{s, +, \times\},\ \{<\},\ \mathrm{ar}\Big)$$

The **signature of arithmetic**, with $\mathrm{ar}(s) = 1$ and $\mathrm{ar}(+) = \mathrm{ar}(\times) = \mathrm{ar}(<) = 2$; s is read "the successor of", the number after.

- The numerals 0, $s(0)$, $s(s(0))$, … are one name for each number; $x + s(0)$ names the number after whatever x is.
- $x < s(x)$ says that x is less than its successor; $\forall x\, \exists y\, (x < y)$ that every number has a bigger one; $\neg \exists x\, (x < 0)$ that nothing is less than zero.

6.1 · SYNTAX

Remark 6.6: reading the conventions

WRITTEN	READ AS
$\neg p \wedge q$	$(\neg p) \wedge q$, not $\neg(p \wedge q)$
$p \wedge q \to r$	$((p \wedge q) \to r)$
$p \to q \to r$	$(p \to (q \to r))$
$\forall x\, \mathsf{Eng}(x) \to \mathsf{Mgr}(x)$	$(\forall x\, \mathsf{Eng}(x)) \to \mathsf{Mgr}(x)$

$$\forall x\, (\mathsf{Eng}(x) \to \mathsf{Mgr}(\mathsf{boss}(x)))$$

"every engineer reports to a manager"

6.1 · SYNTAX

Remark 6.7: “every” goes with $\rightarrow$, “some” with $\wedge$

EVERY P …

Restricts a universal quantifier, with $\rightarrow$: for every x, *if* x is a P, then …

SOME P …

Restricts an existential quantifier, with $\wedge$: some x is a P *and* …

Universal with $\rightarrow$, existential with $\wedge$: reliable, but not infallible.

6.1 · SYNTAX

Remark 6.7: with the other connective

$$\forall x\,(\mathsf{Eng}(x) \wedge \mathsf{Mgr}(\mathsf{boss}(x)))$$

"everyone is an engineer, and reports to a manager"

False of any firm that employs somebody else.

$$\exists x\,(\mathsf{Mgr}(x) \to \mathsf{boss}(x) \neq x)$$

"someone, if a manager, has a boss other than themselves"

Made true by anyone who is not a manager.

6.1 · SYNTAX

Which variables still need a value?

$$\mathsf{Richer}(\mathsf{boss}(x), x) \qquad \text{and} \qquad \forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$$

- The first is true or false according to who x is.
- The second makes one claim about the whole domain: everyone earns less than their boss.

The next definition separates the variables whose values must be supplied from those a quantifier has taken charge of.

6.1 · SYNTAX

From anybody to somebody

- Proofs pass from a claim about anybody to the same claim about somebody in particular, as (∀E) does, and back, as (∃I) does.
- The tool is **substitution**: $\mathsf{Richer}(\mathsf{boss}(x), x)$ becomes $\mathsf{Richer}(\mathsf{boss}(\mathsf{ceo}), \mathsf{ceo})$, the claim made of the chief executive.
- The danger: the term put in may have variables of its own, and a quantifier in the formula can **capture** them.

So two definitions: substitution, and the condition that rules capture out.

6.1 · SYNTAX

Definition 6.8: free variables

The free variables of a term or formula are those whose values must be supplied before it names or says anything. First the variables of a term:

$$\mathrm{var}(t) := \begin{cases} \{x\} & \text{if } t \text{ is a variable } x, \\ \emptyset & \text{if } t \text{ is a constant } c, \\ \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_n) & \text{if } t \text{ is } f(t_1, \ldots, t_n). \end{cases}$$

6.1 · SYNTAX

Definition 6.8, continued: free variables of a formula

With $\circ$ any of $\to$, $\wedge$, $\vee$, and Q either quantifier:

$$\mathrm{FV}(\varphi) := \begin{cases} \emptyset & \text{if } \varphi \text{ is } \bot, \\ \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_n) & \text{if } \varphi \text{ is } R(t_1, \ldots, t_n), \\ \mathrm{FV}(\psi) \cup \mathrm{FV}(\chi) & \text{if } \varphi \text{ is } \psi \circ \chi, \\ \mathrm{FV}(\psi) \setminus \{x\} & \text{if } \varphi \text{ is } Q x\, \psi. \end{cases}$$

The last line is the one that matters: a quantifier removes its variable.

6.1 · SYNTAX

Definition 6.8, continued: bound and free

An occurrence of x in φ is **bound** if it lies inside a part of φ of the form $\forall x\,\psi$ or $\exists x\,\psi$, and **free** otherwise. The quantifier **binds** the occurrences of x in ψ.

6.1 · SYNTAX

Definition 6.9: closed term; sentence

A term t is **closed** if $\mathrm{var}(t) = \emptyset$, and a formula φ is a **sentence** if $\mathrm{FV}(\varphi) = \emptyset$.

6.1 · SYNTAX

Definition 6.10: substitution

Substitution puts a term in place of the free occurrences of a variable. Let t be a term and x a variable. For a term u:

$$u[t/x] := \begin{cases} t & \text{if } u \text{ is } x, \\ u & \text{if } u \text{ is a variable } y \neq x \text{ or a constant}, \\ f(u_1[t/x], \dots, u_n[t/x]) & \text{if } u \text{ is } f(u_1, \dots, u_n). \end{cases}$$

6.1 · SYNTAX

Definition 6.10, continued: substitution in a formula

With $\circ$ any of $\to$, $\wedge$, $\vee$, and Q either quantifier:

$$\varphi[t/x] := \begin{cases} \bot & \text{if } \varphi \text{ is } \bot, \\ R(u_1[t/x], \dots, u_n[t/x]) & \text{if } \varphi \text{ is } R(u_1, \dots, u_n), \\ \psi[t/x] \circ \chi[t/x] & \text{if } \varphi \text{ is } \psi \circ \chi, \\ Qx\,\psi & \text{if } \varphi \text{ is } Qx\,\psi, \\ Qy\,\psi[t/x] & \text{if } \varphi \text{ is } Qy\,\psi \text{ with } y \neq x. \end{cases}$$

6.1 · SYNTAX

Definition 6.11: substitutable

A term is substitutable for a variable when putting it in place of the variable lets no quantifier capture a variable of the term.

The term t is **substitutable for x in φ** if no free occurrence of x in φ lies inside a part of φ of the form $\forall y\,\psi$ or $\exists y\,\psi$ with $y \in \mathrm{var}(t)$.

We write $\varphi[t/x]$ only when t is substitutable for x in φ. A closed term always is.

Example 6.12: free and bound

- ceo and $\mathsf{boss}(\mathsf{boss}(\mathsf{ceo}))$ are closed terms; $\mathsf{boss}(x)$ is not.
- $\mathsf{Richer}(\mathsf{boss}(x), x)$ is not a sentence: its one free variable is x. $\forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$ is, since the last line of the definition of FV removes x.

$$\mathsf{Eng}(x) \to \forall x\, \mathsf{Mgr}(x)$$

"if x is an engineer, then everyone is a manager"

The first occurrence of x is free, the other two are bound: not a sentence.

Example 6.12: substitution

Let φ be $x \neq \mathsf{ceo} \to \mathsf{Richer}(\mathsf{boss}(x), x)$. Then $\varphi[\mathsf{boss}(\mathsf{ceo})/x]$ is

$$\mathsf{boss}(\mathsf{ceo}) \neq \mathsf{ceo} \to \mathsf{Richer}(\mathsf{boss}(\mathsf{boss}(\mathsf{ceo})), \mathsf{boss}(\mathsf{ceo}))$$

"if the chief executive's boss is not the chief executive, that boss earns less than their own boss"

The claim made of x is now made of the chief executive's boss.

Example 6.12: capture

Let φ be $\exists y\, \mathsf{Richer}(y, x)$, "someone earns more than x", and t be $\mathsf{boss}(y)$. The recursion gives

$$\exists y\, \mathsf{Richer}(y, \mathsf{boss}(y))$$

"someone earns more than their own boss"

- A different claim altogether: the y of the term has been **captured**.
- The condition forbids it: the free x lies inside $\exists y\, \mathsf{Richer}(y, x)$, and $y \in \mathrm{var}(\mathsf{boss}(y))$.

6.1 · SYNTAX

Nothing here says what is true

- The sentences of Figure 6.2 look as if they speak about one firm, and we have read them so.
- But $\forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$ mentions no structure: every structure for its signature reads it.

To say when a sentence is true, we must first say what it is true of.

SECTION 6.2

Structures

What a sentence is true of.

6.2 · STRUCTURES

Five systems

SYSTEM	THINGS	RELATIONS	FUNCTIONS
the company	five people	is an engineer; is a manager; earns more than	the boss of
counting numbers	$0, 1, 2, \ldots$	is less than	successor, $+$, $\times$
server network	four machines	is linked to	none
a program	its states	is an error state	the next state
truth values	true and false	none	and, or, not

Figure 6.3: five systems.

- Nothing in common as subject matter; the same **three pieces**: a collection of things, relations among them, functions on them.
- And a few things with names of their own: *the* chief executive, *the* number 0, *the* initial state.

6.2 · STRUCTURES

A system, seen through a signature

- What a sentence is true of has two parts: what the symbols of the signature are taken to mean, and among what things.
- The relations, the functions and the named things of a system are exactly what a signature has symbols for.

A structure is a system seen through a signature: which things there are, and what each symbol means among them.

6.2 · STRUCTURES

Definition 6.13: structure

A structure for a signature says which things there are and what each symbol means.

A **Σ-structure** is a pair $\mathfrak{M} = (\mathbb{D}, I)$: a *non-empty* set $\mathbb{D}$, the **domain**, and an **interpretation** I on $\mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$ with

1. $I(c) \in \mathbb{D}$ for every constant symbol c;
2. $I(f)$ a function from $\mathbb{D}^n$ to $\mathbb{D}$ for every f with $\mathrm{ar}(f) = n$;
3. $I(R) \subseteq \mathbb{D}^n$ for every R with $\mathrm{ar}(R) = n$.

We write $[\![c]\!]^{\mathfrak{M}}$, $[\![f]\!]^{\mathfrak{M}}$, $[\![R]\!]^{\mathfrak{M}}$ for $I(c)$, $I(f)$, $I(R)$. Here $\mathbb{D}^0 = \{()\}$, whose one element is the empty tuple.

6.2 · STRUCTURES

A structure is a database

- A relation with n places is a table with n columns and one row for each tuple of which it holds.
- So a structure is a database: one table for each relation symbol, one lookup table for each function symbol, and a few named rows.

No accident: Codd's relational model of data (1970), on which SQL rests, treats a database as a finite structure, and its query languages are first-order logic in other notation.

Example 6.14: Tuesday's table as a structure

$$\mathbb{D}_{\mathrm{C}} = \{\text{Thorne}, \text{Finch}, \text{Pratt}, \text{Bell}, \text{Cole}\}, \qquad [\![\mathsf{ceo}]\!]^{\mathfrak{C}} = \text{Thorne},$$
$$[\![\mathsf{boss}]\!]^{\mathfrak{C}} = \{\text{Thorne} \mapsto \text{Thorne},\ \text{Finch} \mapsto \text{Thorne},\ \text{Pratt} \mapsto \text{Thorne},$$
$$\text{Bell} \mapsto \text{Finch},\ \text{Cole} \mapsto \text{Finch}\},$$
$$[\![\mathsf{Eng}]\!]^{\mathfrak{C}} = \{\text{Bell}, \text{Cole}\}, \qquad [\![\mathsf{Mgr}]\!]^{\mathfrak{C}} = \{\text{Thorne}, \text{Finch}\},$$
$$[\![=]\!]^{\mathfrak{C}} = \{(d, d) : d \in \mathbb{D}_{\mathrm{C}}\}, \qquad [\![\mathsf{Richer}]\!]^{\mathfrak{C}} = \{(d, e) : sal(d) > sal(e)\}$$

sal is the salary column of Figure 6.1. The structure is $\mathfrak{C} = (\mathbb{D}_{\mathrm{C}}, I_{\mathrm{C}})$.

Example 6.14: what the structure keeps

- The salaries all differ, so exactly half of the twenty ordered pairs of different people are in $[\![\mathsf{Richer}]\!]^{\mathfrak{C}}$: ten pairs, from (Thorne, Finch) down to (Bell, Cole).
- *sal* itself is **not** part of the structure. The signature has no symbol for a salary, so the structure keeps only what the symbols can ask about: who earns more than whom.
- $=$ is interpreted as identity. Nothing in Definition 6.13 demands it; every structure for the company signature that we describe does it.

6.2 · STRUCTURES

Example 6.14: the picture

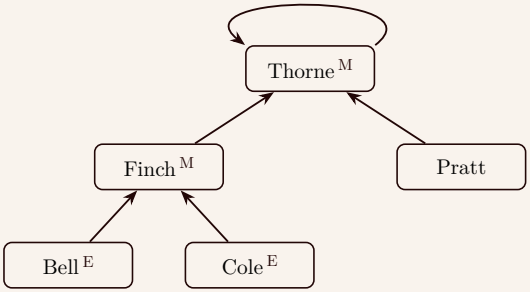


Figure 6.4: the company $\mathfrak{C}$. An arrow from each person to their boss; M a manager, E an engineer.

THE LOOP

Thorne's arrow to herself is the row Harriet's first query returned.

6.2 · STRUCTURES

Listed, or described

A FINITE DOMAIN

The structure can be listed, table by table, as a personnel table can.

AN INFINITE DOMAIN

The structure can be described but not printed, and how to describe one exactly is a delicate matter.

The counting numbers are of the second kind.

6.2 · STRUCTURES

Example 6.15: the counting numbers

The counting numbers have an infinite domain. In the signature of arithmetic of Example 6.5, they are the structure

$\mathfrak{N} = (\mathbb{N}, I_{\mathrm{N}})$, traditionally called the **standard model of arithmetic**: $\mathbb{N} = \{0, 1, 2, ...\}$;

- $[\![0]\!]^{\mathfrak{N}}$ is the number zero, and $[\![s]\!]^{\mathfrak{N}}(n) = n + 1$;
- $[\![+]\!]^{\mathfrak{N}}$ and $[\![\times]\!]^{\mathfrak{N}}$ are addition and multiplication;
- $[\![<]\!]^{\mathfrak{N}} = \{(m, n) \in \mathbb{N}^2 : m \text{ is less than } n\}$.

Example 6.16: the other three systems

- The server network: a domain of four machines and one two-place relation.
- The program: a domain of states, a constant for the initial state, a one-place relation for the error states and a one-place function for the next state.
- The truth values: a domain of two elements with three functions and no relations.

6.2 · STRUCTURES

Remark 6.17: non-empty domains

$$\forall x\, \mathsf{Mgr}(x) \to \exists x\, \mathsf{Mgr}(x)$$

"if everything is a manager, then something is a manager"

- In an empty structure everything is a manager, since there is nothing that is not; and yet nothing is a manager. The sentence would be false.
- Almost every textbook requires the domain to be non-empty, and so do we. The requirement rules this out.

6.2 · STRUCTURES

Remark 6.18: total functions

- The functions of a structure are **total**: an element of the domain for every input from the domain.
- That is what makes every term name something: the boss of anybody's boss is always somebody.
- A real system need not be so obliging. Where a value may be missing, a relation such as *reports to* can stand in for the function *the boss of*.

Example 6.19: a near miss

CASE: THORNE'S BOSS LEFT BLANK

Suppose the table left Thorne's boss blank, since the chief executive has none. A sensible way to run a company.

- Not a structure: $[\![\mathsf{boss}]\!]^{\mathfrak{E}}$ would have a hole at Thorne, and a lookup table with a hole in it is not a function.
- Somebody had to choose a value for Thorne, and choosing Thorne herself is the reason Harriet's first query returned a row.

SECTION 6.3

Satisfaction

A sentence and a structure: out comes a verdict.

6.3 · SATISFACTION

Writing down what the words mean

- A conjunction is true when both conjuncts are; a disjunction when at least one disjunct is.
- "For every x" is true when what follows holds whichever element x stands for.
- The structure enters only at the atomic formulas: $R(t_1, \ldots, t_n)$ is true when the things the terms name stand in the relation R names.

We already know what the logical words mean. The definition writes it down, one kind of formula at a time.

6.3 · SATISFACTION

One thing is still missing

$$\mathsf{Richer}(\mathsf{boss}(x), x)$$

"the boss of x earns more than x"

- A structure does not say what the variables stand for, and a formula with free variables is true or false only once they have values.
- So three definitions, one after another: what the variables stand for, an **assignment**; what a term names, its **value**; when a formula is true, **satisfaction**.

6.3 · SATISFACTION

Definition 6.20: assignment

An assignment says what each variable stands for.

Let $\mathfrak{M}$ be a structure with domain $\mathbb{D}$. An **assignment** into $\mathfrak{M}$ is a function $s : \mathrm{Var} \to \mathbb{D}$. For $x \in \mathrm{Var}$ and $d \in \mathbb{D}$, the assignment $s[d/x]$, "s with d for x", is

$$s[d/x](y) := \begin{cases} d & \text{if } y \text{ is } x, \\ s(y) & \text{if } y \neq x. \end{cases}$$

6.3 · SATISFACTION

Definition 6.21: value of a term

Together, a structure and an assignment settle what every term names.

Let s be an assignment into $\mathfrak{M}$. The **value** $\mathrm{Val}_s^{\mathfrak{M}}(t) \in \mathbb{D}$ of a term t in $\mathfrak{M}$ under s is

$$\mathrm{Val}_s^{\mathfrak{M}}(t) := \begin{cases} s(x) & \text{if } t \text{ is a variable } x, \\ [\![c]\!]^{\mathfrak{M}} & \text{if } t \text{ is a constant } c, \\ [\![f]\!]^{\mathfrak{M}}(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)) & \text{if } t \text{ is } f(t_1, \dots, t_n). \end{cases}$$

We write $\mathrm{Val}_s(t)$ when the structure is clear.

6.3 · SATISFACTION

Definition 6.22: satisfaction

Satisfaction says when a formula is true in a structure under an assignment.

Let s be an assignment into $\mathfrak{M}$. The relation $\mathfrak{M}, s \vDash \varphi$, read "$\varphi$ is **true in $\mathfrak{M}$ under s**" or "$\mathfrak{M}, s$ **satisfies** φ", is defined by recursion on the shape of φ by the clauses of Figure 6.5, one for each kind of formula.

$\mathfrak{M}, s \nvDash \varphi$ means that $\mathfrak{M}, s \vDash \varphi$ does not hold, and "or" is inclusive.

Figure 6.5: satisfaction

$$
\begin{array}{lll}
\mathfrak{M}, s \vDash R(t_1, \dots, t_n) & \text{iff} & (\mathrm{Val}_s(t_1), \dots, \mathrm{Val}_s(t_n)) \in [\![R]\!]^{\mathfrak{M}} \\
\mathfrak{M}, s \nvDash \bot & & \\
\mathfrak{M}, s \vDash \psi \wedge \chi & \text{iff} & \mathfrak{M}, s \vDash \psi \text{ and } \mathfrak{M}, s \vDash \chi \\
\mathfrak{M}, s \vDash \psi \vee \chi & \text{iff} & \mathfrak{M}, s \vDash \psi \text{ or } \mathfrak{M}, s \vDash \chi \\
\mathfrak{M}, s \vDash \psi \to \chi & \text{iff} & \mathfrak{M}, s \nvDash \psi \text{ or } \mathfrak{M}, s \vDash \chi \\
\mathfrak{M}, s \vDash \forall x\, \psi & \text{iff} & \mathfrak{M}, s[d/x] \vDash \psi \text{ for every } d \in \mathbb{D} \\
\mathfrak{M}, s \vDash \exists x\, \psi & \text{iff} & \mathfrak{M}, s[d/x] \vDash \psi \text{ for some } d \in \mathbb{D}
\end{array}
$$

6.3 · SATISFACTION

Reading the clauses

- The clause for $\bot$: it is true nowhere.
- $\neg\varphi$ is $\varphi \to \bot$, so the clause for $\to$ yields one for negation:

$$\mathfrak{M}, s \vDash \neg\varphi \quad \text{iff} \quad \mathfrak{M}, s \nvDash \varphi$$

"$\neg\varphi$ is true exactly when φ is false"

6.3 · SATISFACTION

The quantifier clauses

- To decide $\forall x\, \psi$, look at ψ with x standing for each element in turn, every other variable keeping its value. That is exactly $s[d/x]$.
- Whatever s gave x before does not matter: the quantifier has taken charge of x, as a local variable hides an outer one of the same name.
- The job falls to the assignment because a constant names one thing once and for all, while a variable can stand for one element now and another a moment later.

6.3 · SATISFACTION

On a finite structure, an algorithm

FINITE DOMAIN

Unfold the sentence clause by clause. Each quantifier clause runs through the domain: k nested quantifiers over n elements need about n^k cases.

INFINITE DOMAIN

The quantifier clauses ask about infinitely many elements. The definition still says what truth *is*, but no longer how to find it out.

6.3 · SATISFACTION

Propositional letters

A propositional letter p is a relation symbol with no places, so $[\![p]\!]^{\mathfrak{M}} \subseteq \mathbb{D}^0 = \{()\}$, a set with one element.

- So $[\![p]\!]^{\mathfrak{M}}$ is empty or all of $\mathbb{D}^0$, and the atomic clause makes p false or true. The assignment plays no part: p has no terms.
- The clauses for $\bot$ and the connectives then compute the truth of a formula from the truth of its parts.

For a formula built from letters, whether $\mathfrak{M}, s \vDash \varphi$ depends only on which letters $\mathfrak{M}$ makes true.

6.3 · SATISFACTION

Valuations

- For such formulas a structure amounts to a choice of true or false for each letter: a **valuation**.
- Propositional logic is the part of first-order logic in which every relation symbol has no places.

Its structures are, in effect, the rows of a truth table.

6.3 · SATISFACTION

A sentence should not need an assignment

- The definition asks for an assignment even for a sentence, which has no free variables.
- If its truth depended on one, “true in this structure” would mean nothing until somebody chose an assignment.
- And whether everybody earns less than their boss would have no answer.

6.3 · SATISFACTION

Lemma 6.23: relevance

LEMMA 6.23 · RELEVANCE

Let s and s' be assignments into $\mathfrak{M}$ with $s(x) = s'(x)$ for every $x \in \mathrm{FV}(\varphi)$. Then $\mathfrak{M}, s \vDash \varphi$ iff $\mathfrak{M}, s' \vDash \varphi$.

"a formula cares only about the variables it leaves free"

- *Atomic formulas.* The value of a term depends only on its variables.
- *Connectives.* Their clauses pass the claim straight down.
- $\forall x\,\psi$, $\exists x\,\psi$. $\mathrm{FV}(\psi) \subseteq \mathrm{FV}(\forall x\,\psi) \cup \{x\}$, where $s[d/x]$ and $s'[d/x]$ agree; use the induction hypothesis for every d.

6.3 · SATISFACTION

True in a structure

$$\mathfrak{M} \vDash \varphi$$

"the sentence φ is true in the structure $\mathfrak{M}$"

when $\mathfrak{M}, s \vDash \varphi$ for one assignment s, and so, by relevance, for all.

6.3 · SATISFACTION

Lemma 6.24: substitution

Substitution passes from a claim about anybody to the same claim about somebody. That is safe only if $\varphi[t/x]$ says about the thing t names what φ says about x.

LEMMA 6.24 · SUBSTITUTION

If t is substitutable for x in φ, then $\mathfrak{M}, s \vDash \varphi[t/x]$ iff $\mathfrak{M}, s[\mathrm{Val}_s(t)/x] \vDash \varphi$.

"putting the term t in for x has the same effect as letting x stand for whatever t names"

Without the proviso it fails (the capture of Example 6.12). The proof, by induction on φ, is long and unilluminating: omitted.

6.3 · SATISFACTION

Bivalence

- The clause for negation makes exactly one of φ and $\neg\varphi$ true.
- So, by the clause for disjunction, $\varphi \vee \neg\varphi$ is true in every structure under every assignment, whatever φ says.

Bivalence: every formula has exactly one of two truth values in a structure under an assignment.

6.3 · SATISFACTION

Remark 6.25: unnamed elements

Why not define truth by substitution: $\forall x\, \psi$ true when $\psi[t/x]$ is true for every closed term t?

- Take the real numbers, with symbols 0, s, $<$ read as zero, adding one and less than.
- Every closed term, 0, $s(0)$, $s(s(0))$, …, names a natural number, and none lies strictly between 0 and 1. Yet

$$\exists x\, (0 < x \wedge x < s(0))$$

"some number lies strictly between 0 and 1"

is true. A domain can hold elements no closed term names; an assignment reaches them all.

6.3 · SATISFACTION

Remark 6.26: one sentence, several structures

$$\forall x \, \exists y \, R(x, y)$$

"everything is R-related to something"

STRUCTURE	R READ AS	VERDICT
the company	earns more than	false: Cole earns more than nobody
the counting numbers	$<$	true: $n + 1$ is a witness for n
three servers in a ring	is linked to	true
three servers in a line, a to b to c	is linked to	false at c

Truth belongs to the pair of a sentence and a structure, not to the sentence alone.

6.3 · SATISFACTION

Example 6.27: the boss's boss

In $\mathfrak{C}$, under an assignment s with $s(x) = \text{Cole}$:

$$\text{Val}_s(\mathsf{boss}(\mathsf{boss}(x))) = [\![\mathsf{boss}]\!]^{\mathfrak{C}}([\![\mathsf{boss}]\!]^{\mathfrak{C}}(\text{Cole})) = [\![\mathsf{boss}]\!]^{\mathfrak{C}}(\text{Finch}) = \text{Thorne}$$

"Cole's boss's boss is Thorne"

The closed term $\mathsf{boss}(\mathsf{boss}(\mathsf{ceo}))$ *also has the value Thorne, and under every assignment.*

Example 6.27: evaluating a universal sentence

$$\forall x\,(x \neq \mathsf{ceo} \rightarrow \mathsf{Richer}(\mathsf{boss}(x), x))$$

"everyone except the chief executive earns less than their boss"

d	$x \neq \mathsf{ceo}$?	BOSS OF d	$\mathsf{Richer}(\mathsf{boss}(x), x)$?	THE IMPLICATION
Thorne	no	Thorne	no ($200 \not> 200$)	true (antecedent false)
Finch	yes	Thorne	yes ($200 > 120$)	true
Pratt	yes	Thorne	yes ($200 > 95$)	true
Bell	yes	Finch	yes ($120 > 90$)	true
Cole	yes	Finch	yes ($120 > 85$)	true

Figure 6.6: one row for each element of the domain.

6.3 · SATISFACTION

Example 6.27: the Thorne row

- Without the antecedent the Thorne row is false, so $\forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$ is false in $\mathfrak{C}$, with Thorne as the counterexample: Harriet's two queries again, now as two applications of a definition.
- Why is the implication *true* at Thorne, when its consequent is false? Read it as a rule: if you are not the chief executive, you earn less than your boss.
- A rule of that form is broken only by someone it applies to. It does not apply to Thorne.

An implication fails only when its antecedent is true and its consequent false.

6.3 · SATISFACTION

Example 6.27: a $\forall\exists$-sentence

$$\forall x\,(\mathsf{Mgr}(x) \to \exists y\,(\mathsf{boss}(y) = x \wedge y \neq x))$$

"every manager manages someone other than themselves"

- For the three people who are not managers, the implication is true at once.
- For Thorne, the clause for $\exists$ asks for one d whose boss is Thorne and who is not Thorne: Finch will do. For Finch, Bell will do.

True in $\mathfrak{C}$. For the universal quantifier we examined every row; for the existential one we produced one witness.

6.3 · SATISFACTION

Example 6.28: propositional logic

Two letters, four valuations, one per row. The clauses compute the column of each connective from the columns of its parts.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

- The column for $\neg p$ comes from the clause for $\to$, with $\bot$, false in every row, as consequent.
- The column for $p \to q$ is false in one row only: p true and q false.

6.3 · SATISFACTION

Example 6.28: a longer formula

$$((p \to q) \wedge \neg q) \to \neg p$$

"if p implies q, and q is false, then p is false"

p	q	$p \to q$	$\neg q$	$(p \to q) \wedge \neg q$	$\neg p$	THE WHOLE
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

True in every row. Every structure gives p and q the values of one row, so the formula is true in every structure under every assignment.

6.3 · SATISFACTION

Example 6.28: one row is enough

$$(p \to q) \to (q \to p)$$

"if p implies q, then q implies p"

- False in the third row: $p \to q$ is true there and $q \to p$ false.
- A structure that makes p false and q true falsifies it, and four rows were enough to find one.

With n letters there are 2^n rows to check.

SECTION 6.4

Entailment

True in every structure a theory allows.

6.4 · ENTAILMENT

Requirements, not a system

- Satisfaction answers a question about one structure and one assignment: is this formula true *here*?
- Often there is no particular system in view, only requirements that every acceptable system must meet.
- We want to know what holds in every system that meets them: whether everybody earns less than their boss in every table a personnel system could ever accept.

6.4 · ENTAILMENT

Definition 6.29: theory; model

A theory is a set of sentences, and its models are the structures in which all its sentences are true.

A Σ-**theory** is a set of Σ-sentences. A Σ-structure $\mathfrak{M}$ is a **model of** a Σ-theory Θ, or **satisfies** Θ, written $\mathfrak{M} \models \Theta$, if $\mathfrak{M} \models \psi$ for every $\psi \in \Theta$.

For a set Γ of formulas, $\mathfrak{M}, s \models \Gamma$ means $\mathfrak{M}, s \models \psi$ for every $\psi \in \Gamma$. For a theory the assignment makes no difference, by relevance.

6.4 · ENTAILMENT

Definition 6.30: entailment

A set of formulas entails a formula if the formula is true wherever all the members of the set are; a formula entailed by the empty set is valid.

Let Γ be a set of Σ-formulas and φ a Σ-formula. Γ **entails** φ, written $\Gamma \models \varphi$, if $\mathfrak{M}, s \models \varphi$ for every Σ-structure $\mathfrak{M}$ and every assignment s into $\mathfrak{M}$ with $\mathfrak{M}, s \models \Gamma$.

φ is **valid**, written $\models \varphi$, if $\emptyset \models \varphi$: true in every structure under every assignment.

For a theory Θ and a sentence φ: $\Theta \models \varphi$ says φ is true in every model of Θ.

6.4 · ENTAILMENT

Three semantic notions

NOTION	MEANS	EXAMINED
$\mathfrak{M}, s \vDash \varphi$	φ is true in $\mathfrak{M}$ under s	one $\mathfrak{M}, s$
$\Gamma \vDash \varphi$	$\mathfrak{M}, s \vDash \varphi$ for every $\mathfrak{M}, s$ with $\mathfrak{M}, s \vDash \Gamma$	those satisfying Γ
$\vDash \varphi$	$\mathfrak{M}, s \vDash \varphi$ for every $\mathfrak{M}, s$	all

Figure 6.7: three semantic notions.

Validity is entailment by the empty theory, and it is rarely what we want: "nobody but the person at the top is their own boss" holds, if it holds, because of the theory.

6.4 · ENTAILMENT

Not on an equal footing

A NON-ENTAILMENT

Comes with a certificate, a structure, which anyone can check by evaluating sentences in it.

AN ENTAILMENT

Comes with an argument, "let $\mathfrak{M}$ be a model of the theory and d an element": convincing, but it speaks of every model of the theory, of every size, and no loop runs over all of those.

The two examples at the end of the section each show one of each.

Remark 6.31: structures and models

- A structure is a mathematical object. A model is always a model **of** something: the same structure may be a model of one theory and not of another.
- A structure with an assignment gets no name of its own: we write $\mathfrak{M}, s$, and read $\mathfrak{M}, s \vDash \varphi$ as "φ is true in $\mathfrak{M}$ under s".
- English also uses *model* the other way round: a physicist's equations model nature. Here the world is the model, and the sentences describing it are the theory.

6.4 · ENTAILMENT

Remark 6.32: one symbol, three jobs

What stands on the left of $\models$ says which job it is doing.

ON THE LEFT	THE CLAIM
a structure and an assignment	$\mathfrak{M}, s \models \varphi$: true in $\mathfrak{M}$ under s; $\mathfrak{M}, s \models \Gamma$: every member is
a structure alone	$\mathfrak{M} \models \varphi$: the sentence is true in it; $\mathfrak{M} \models \Theta$: a model of Θ
a set of formulas, or nothing	$\Gamma \models \varphi$, $\models \varphi$: every structure, every assignment

$\mathfrak{M} \models \varphi$ is a fact about one structure; $\{\psi\} \models \varphi$ is a fact about all of them.

Remark 6.33: the conditional of science

"Implies" is ambiguous in English. The clause for $\to$ fixes one reading: the *law-like* indicative conditional, the "if" in which science states its laws.

> If a body is in motion, then it stays in motion unless an external force acts on it.
>
> Newton's first law.

- Does the law hold of a body at rest? Of course: a body at rest is a case it does not apply to, and a law does not become false by not applying.
- It is broken only by a case it does apply to: a body in motion that stops with no force acting on it.

Remark 6.33: two odd validities

Read this way, a conditional is false only when its antecedent is true and its consequent false: the clause for $\to$. Followed through:

$$\neg\varphi \to (\varphi \to \psi) \qquad \text{and} \qquad \psi \to (\varphi \to \psi)$$

"a false claim implies anything; a true claim is implied by anything"

- A case where the antecedent fails breaks no law, whatever the consequent says; nor does a case where the consequent holds.
- So "if $2 + 2 = 5$, then $1 + 1 = 3$" is true, and "if n is even, then n^2 is even" holds of every n, odd ones included.

6.4 · ENTAILMENT

Remark 6.33: not every “if”

THE CONDITIONAL OF CONVERSATION

Often promises a connection between its two halves.

THE SUBJUNCTIVE

“If I had left earlier, I would have caught the bus”: the antecedent is known to be false, and the claim is still substantial.

No clause that decides $\varphi \to \psi$ from the truth values of φ and ψ alone could capture those, and satisfaction needs exactly that.

Williamson (2020) defends the indicative conditional of English as the material one.

6.4 · ENTAILMENT

Equality needs a theory of its own

- A structure may interpret $=$ as any set of pairs.
- A system that compares names relies on two facts about identity: everything is equal to itself, and equals may be put for equals.

Written as sentences, these facts are the equality axioms.

Remark 6.34: equality axioms

The equality axioms of a signature say, in its own symbols, that $=$ behaves as identity.

Let Σ contain the two-place relation symbol $=$. The **equality axioms** Eq_Σ are the Σ-sentences

(Eq1) $\forall x\,(x = x)$;
(Eq2) $\forall z_1 \cdots \forall z_k\,(x = y \to (\varphi \to \varphi[y/x]))$, for every Σ-formula φ and all variables x, y with y substitutable for x in φ, where $z_1, \ldots, z_k$ are the free variables of $x = y \to (\varphi \to \varphi[y/x])$.

No symmetry, no transitivity: both follow, since (Eq2) allows every φ (Exercise 6.8).

Remark 6.34, continued: normal structures

A structure that interprets $=$ as identity is **normal**.

- Every normal structure satisfies Eq_Σ.
- (Eq1) holds because everything is identical to itself.
- Each instance of (Eq2) holds by the substitution lemma: under an assignment that gives x and y the same value, φ and $\varphi[y/x]$ have the same truth value.

Remark 6.35: equality as a logical symbol

- Many presentations make $=$ logical, with rules of its own: ($=$I) concludes $t = t$ from nothing; ($=$E) concludes $\varphi[t_2/x]$ from $t_1 = t_2$ and $\varphi[t_1/x]$.
- The two treatments derive the same formulas from the same hypotheses: each axiom is derivable with the rules, and each use of a rule is an instance of (Eq1) or (Eq2) followed by ($\forall$E) and ($\to$E).
- Keeping $=$ a relation symbol leaves the logic, and every proof about it, unchanged.

6.4 · ENTAILMENT

Example 6.36: valid, and not valid

$$\forall x\, \mathsf{Eng}(x) \to \mathsf{Eng}(\mathsf{ceo})$$

"if everyone is an engineer, so is the chief executive"

Valid: in every structure, if every element is in $[\![\mathsf{Eng}]\!]^{\mathfrak{M}}$, so is $[\![\mathsf{ceo}]\!]^{\mathfrak{M}}$.

$$\forall x\, \mathsf{Richer}(\mathsf{boss}(x), x)$$

"everyone earns less than their boss"

Not valid, since $\mathfrak{C}$ falsifies it. True in the counting numbers, with ceo as 0, boss as s, Richer as $>$; true in no finite company whose Richer comes from salaries.

Example 6.36: true here, not valid

$$\exists x\, \mathsf{Eng}(x)$$

"somebody is an engineer"

True in $\mathfrak{C}$, and false in every company without engineers.

A sentence true in $\mathfrak{C}$ need not be valid, however natural it sounds.

Example 6.37: the theory of the personnel system

CASE: TWO RULES

Harriet's personnel system enforces, among others, two rules that can be written in the company signature.

$$\mathrm{R}_1 : \ \forall x \, \neg \mathsf{Richer}(x, x) \qquad \mathrm{R}_2 : \ \forall x \, (x \neq \mathsf{ceo} \to \mathsf{Richer}(\mathsf{boss}(x), x))$$

"nobody earns more than themselves; everyone but the chief executive earns less than their boss"

The system also relies, without saying so, on Eq_{Σ_C}. One instance of (Eq2), from $\mathsf{Richer}(u, w)$ with u, v for x, y:

$$\mathrm{E}_{\mathsf{Richer}} : \ \forall u \, \forall v \, \forall w \, (u = v \to (\mathsf{Richer}(u, w) \to \mathsf{Richer}(v, w)))$$

"whoever equals someone who earns more than w also earns more than w"

Example 6.37: the theory Θ

Θ is the set consisting of R_1, R_2 and $\mathrm{Eq}_{\Sigma_\mathrm{C}}$: the theory of the personnel system, in the company signature.

- A normal structure satisfies Θ exactly when both rules hold in it: exactly when it is a table the system would accept.
- $\Theta \models \varphi$ says that φ holds in every state the system can be in. That is what the finance director asked.

6.4 · ENTAILMENT

Example 6.37: an entailment

Sharpened a little, the finance director's question: does the theory guarantee that nobody but the chief executive is her own boss?

$$\Theta \models \forall x\,(x \neq \mathsf{ceo} \rightarrow \mathsf{boss}(x) \neq x)$$

"in every model of the theory, nobody but the chief executive is their own boss"

It does. The argument speaks of an arbitrary model and an arbitrary element.

Example 6.37: the argument

Let $\mathfrak{M} \models \Theta$, and d an element with $(d, [\![\mathsf{ceo}]\!]^{\mathfrak{M}}) \notin [\![=]\!]^{\mathfrak{M}}$. Suppose $([\![\mathsf{boss}]\!]^{\mathfrak{M}}(d), d) \in [\![=]\!]^{\mathfrak{M}}$. Then

$$\begin{aligned} &([\![\mathsf{boss}]\!]^{\mathfrak{M}}(d), d) \in [\![\mathsf{Richer}]\!]^{\mathfrak{M}} && \text{by } \mathrm{R}_2, \\ &(d, d) \in [\![\mathsf{Richer}]\!]^{\mathfrak{M}} && \text{by } \mathrm{E}_{\mathsf{Richer}}, \text{ with } [\![\mathsf{boss}]\!]^{\mathfrak{M}}(d),\ d,\ d \text{ for } u,\ v,\ w, \end{aligned}$$

which contradicts R_1. So $\mathsf{boss}(x) \neq x$ is true under $s[d/x]$. For d the chief executive the implication is true by its clause. Since $\mathfrak{M}$ and d were arbitrary, the entailment holds.

6.4 · ENTAILMENT

Example 6.37: a non-entailment

Does the theory guarantee that no engineer is a manager? To show that an entailment *fails*, one structure suffices.

$$\Theta \models \forall x\,(\mathsf{Eng}(x) \to \neg\mathsf{Mgr}(x))\ ?$$

THE ONE-PERSON COMPANY

Normal, domain {Ada}: ceo names Ada, boss sends Ada to Ada, Eng and Mgr are both {Ada}, Richer is empty.

THE CHECKS

R_1: Richer is empty. R_2: its antecedent is never true. Eq: the structure is normal. Ada is an engineer who is a manager.

6.4 · ENTAILMENT

Example 6.38: a theory of order

In the signature of arithmetic, let $\Theta_<$ be the theory with the two sentences

$$\mathrm{O}_1 : \ \forall x \neg(x < x) \qquad \mathrm{O}_2 : \ \forall x \, \forall y \, \forall z \, (x < y \wedge y < z \to x < z)$$

"nothing is less than itself; "less than" passes along chains"

The counting numbers are a model of $\Theta_<$: no number is less than itself, and if $l < m$ and $m < n$ then $l < n$.

Example 6.38: an entailment

$$\Theta_< \models \forall x\, \forall y\, (x < y \to \neg(y < x))$$

"“less than” never runs both ways"

Let $\mathfrak{M} \models \Theta_<$, and d, e elements with $(d, e) \in [\![<]\!]^{\mathfrak{M}}$. Suppose $(e, d) \in [\![<]\!]^{\mathfrak{M}}$ as well. Then

$$(d, d) \in [\![<]\!]^{\mathfrak{M}} \qquad \text{by } \mathrm{O}_2\text{, with } d\text{, } e\text{, } d \text{ for } x\text{, } y\text{, } z\text{,}$$

which contradicts O_1. So $(e, d) \notin [\![<]\!]^{\mathfrak{M}}$; since $\mathfrak{M}$, d and e were arbitrary, the entailment holds.

Example 6.38: true in $\mathfrak{N}$, not entailed

$$\forall x \, \exists y \, (x < y)$$

"every number has a bigger one"

True in the counting numbers. But $\Theta_<$ does not entail it:

ONE ELEMENT

Domain $\{a\}$: 0 names a, s, $+$ and $\times$ send everything to a, and $<$ is empty. O_1 holds, since nothing is less than anything; O_2 holds, since its antecedent is never true; the sentence is false at a.

True in $\mathfrak{N}$ is one thing; following from a theory that $\mathfrak{N}$ satisfies is another.

SECTION 6.5

Soundness of NJ

Derivations never lead from truth to falsehood.

6.5 · SOUNDNESS OF NJ

A certificate for an entailment

- Listing the structures that satisfy Γ is out of the question: there are infinitely many, and most of them are infinite.
- What we want is a **proof**: a finite object showing that the commitments of Γ yield φ, each step of which a program can check.
- Part 1 supplied one, Gentzen's NJ.

$$\Gamma \vdash_{\mathrm{NJ}} \varphi$$

"some derivation in NJ has conclusion φ and all its open hypotheses in Γ"

6.5 · SOUNDNESS OF NJ

Two notions of consequence

$\Gamma \vDash \varphi$ quantifies over all structures and assignments; $\Gamma \vdash_{\mathrm{NJ}} \varphi$ asks for one finite object. Two connections are possible:

SOUNDNESS

If $\Gamma \vdash_{\mathrm{NJ}} \varphi$, then $\Gamma \vDash \varphi$.

Derivations are safe: none leads from true hypotheses to a false conclusion.

COMPLETENESS

If $\Gamma \vDash \varphi$, then $\Gamma \vdash_{\mathrm{NJ}} \varphi$.

Derivations are enough: whatever follows has one.

6.5 · SOUNDNESS OF NJ

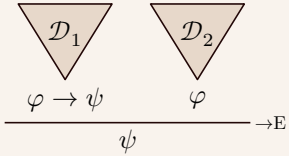
Induction on derivations

- Soundness is a claim about every derivation, and there are infinitely many, of every size.
- A claim about every counting number is proved by induction: show it for 0, show that it passes from n to $n + 1$. Every number is reached from 0 by adding 1.
- Derivations are built the same way, from leaves by rules. A derivation is a single leaf, a hypothesis, or a rule applied to one, two or three smaller derivations, its immediate subderivations.

6.5 · SOUNDNESS OF NJ

Derivations as triangles

Drawn as in Part 1, each smaller derivation is a triangle, with its name inside and its conclusion at the point; the rule is a bar beneath them. What (→E) builds from $\mathcal{D}_1$ and $\mathcal{D}_2$:



Every derivation is reached by starting from leaves and adding one rule at a time.

6.5 · SOUNDNESS OF NJ

The base and the steps

To show that a property holds of *every* derivation, it is enough to show two things:

THE BASE

The property holds of every single leaf.

THE STEPS

For each rule: if the property holds of the immediate subderivations, it holds of the derivation the rule builds from them.

The supposition in a step is the induction hypothesis. *The base gives the leaves; the steps give every derivation with one rule, then two, and so on.*

6.5 · SOUNDNESS OF NJ

Which property?

- Not “the conclusion is true”: a derivation reasons from suppositions, and a supposition may be false, and so may everything drawn from it.
- Not anything about a fixed set Γ: ($\to$I) discharges, so a subderivation may have an open hypothesis that the whole derivation lacks.

(S), FOR A DERIVATION $\mathcal{D}$ WITH CONCLUSION φ

For every structure $\mathfrak{M}$ and every assignment s into $\mathfrak{M}$: if every formula of $\mathrm{OH}(\mathcal{D})$ is true in $\mathfrak{M}$ under s, then so is φ.

“the conclusion of a derivation can be false only where one of its open hypotheses is”

6.5 · SOUNDNESS OF NJ

Theorem 6.39: soundness of NJ

THEOREM 6.39 · SOUNDNESS OF NJ

For every set Γ of formulas and every formula φ, if $\Gamma \vdash_{\mathrm{NJ}} \varphi$ then $\Gamma \models \varphi$.

"what NJ derives from Γ is true wherever Γ is"

Proof. We show (S) for every derivation of NJ, by induction on derivations. The theorem follows: if $\Gamma \vdash_{\mathrm{NJ}} \varphi$, some $\mathcal{D}$ has conclusion φ and $\mathrm{OH}(\mathcal{D}) \subseteq \Gamma$; if $\mathfrak{M}, s \models \Gamma$, every formula of $\mathrm{OH}(\mathcal{D})$ is true, and (S) makes φ true.

6.5 · SOUNDNESS OF NJ

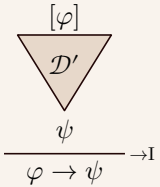
The proof: what each step shows

- In each step, fix $\mathfrak{M}$ and s, suppose every formula of $\mathrm{OH}(\mathcal{D})$ is true, and show that the conclusion is. The induction hypothesis is (S) for the subderivations.
- For the connectives only truth values matter, and the clause for the connective: think of φ, ψ, χ as letters, each simply true or false.
- The quantifier rules look inside a formula, and they need the relevance and substitution lemmas.

The base: a single leaf φ. Its only open hypothesis is φ itself, true by supposition.

6.5 · SOUNDNESS OF NJ

Soundness: (→I)

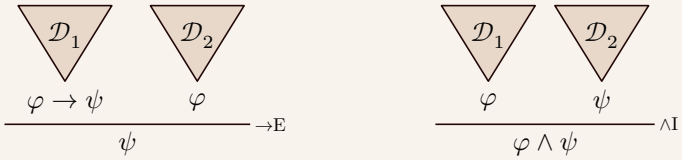


$\mathrm{OH}(\mathcal{D})$ is $\mathrm{OH}(\mathcal{D}')$ with φ removed.

- If φ is false under s, then $\varphi \to \psi$ is true by the clause for $\to$, whether or not ψ is.
- If φ is true, every formula of $\mathrm{OH}(\mathcal{D}')$ is true, since each is φ or in $\mathrm{OH}(\mathcal{D})$. So ψ is true by the induction hypothesis, and so is $\varphi \to \psi$.

6.5 · SOUNDNESS OF NJ

Soundness: ($\to$E) and ($\wedge$I)



In both, $\mathrm{OH}(\mathcal{D})$ is $\mathrm{OH}(\mathcal{D}_1) \cup \mathrm{OH}(\mathcal{D}_2)$, so every open hypothesis of each part is true.

- *($\to E$).* By the induction hypothesis $\varphi \to \psi$ and φ are true. By the clause for $\to$, φ is false or ψ is true; φ is true, so ψ is.
- *($\wedge I$).* By the induction hypothesis φ and ψ are true, and by the clause for $\wedge$ so is $\varphi \wedge \psi$.

6.5 · SOUNDNESS OF NJ

Soundness: ($\wedge$E), ($\vee$I) and ($\bot$E)

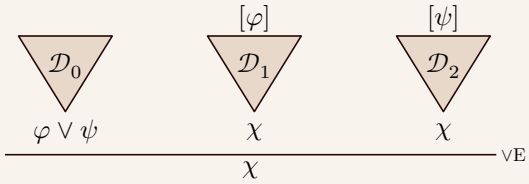
$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \varphi \wedge \psi\end{array}}{\varphi}\,\wedge\mathrm{E} \qquad \dfrac{\begin{array}{c}\mathcal{D}'\\ \varphi\end{array}}{\varphi \vee \psi}\,\vee\mathrm{I} \qquad \dfrac{\begin{array}{c}\mathcal{D}'\\ \bot\end{array}}{\varphi}\,\bot\mathrm{E}$$

In all three, $\mathrm{OH}(\mathcal{D})$ is $\mathrm{OH}(\mathcal{D}')$.

- *($\wedge E$).* $\varphi \wedge \psi$ is true, so both conjuncts are (and likewise with ψ beneath).
- *($\vee I$).* φ is true, so $\varphi \vee \psi$ is (and likewise with ψ in the triangle).
- *($\bot E$).* $\bot$ would be true, which its clause forbids. No $\mathfrak{M}, s$ makes every open hypothesis true, so there is no case to check.

6.5 · SOUNDNESS OF NJ

Soundness: (∨E)



$\mathrm{OH}(\mathcal{D})$ is $\mathrm{OH}(\mathcal{D}_0)$, with $\mathrm{OH}(\mathcal{D}_1)$ without φ and $\mathrm{OH}(\mathcal{D}_2)$ without ψ.

- By the induction hypothesis for $\mathcal{D}_0$, $\varphi \vee \psi$ is true, so φ is true or ψ is.
- If φ is, every formula of $\mathrm{OH}(\mathcal{D}_1)$ is true, and χ is true by the induction hypothesis for $\mathcal{D}_1$. If ψ is, the same with $\mathcal{D}_2$.

6.5 · SOUNDNESS OF NJ

Soundness: (∀E) and (∃I)

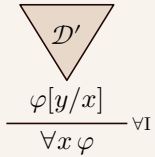
$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \forall x\,\varphi\end{array}}{\varphi[t/x]}\,\forall\mathrm{E} \qquad\qquad \dfrac{\begin{array}{c}\mathcal{D}'\\ \varphi[t/x]\end{array}}{\exists x\,\varphi}\,\exists\mathrm{I}$$

t substitutable for x in φ; $\mathrm{OH}(\mathcal{D})$ is $\mathrm{OH}(\mathcal{D}')$.

- *($\forall E$).* $\forall x\,\varphi$ is true, so φ is true under $s[d/x]$ for every d, in particular $d = \mathrm{Val}_s(t)$; by the substitution lemma, $\varphi[t/x]$ is true under s.
- *($\exists I$).* $\varphi[t/x]$ is true, so by the substitution lemma φ is true under $s[\mathrm{Val}_s(t)/x]$, and $\exists x\,\varphi$ is true by its clause.

6.5 · SOUNDNESS OF NJ

Soundness: (∀I)



y is free in no formula of $\mathrm{OH}(\mathcal{D}')$, which is $\mathrm{OH}(\mathcal{D})$, and not free in $\forall x\,\varphi$. We must show φ true under $s[d/x]$ for every d.

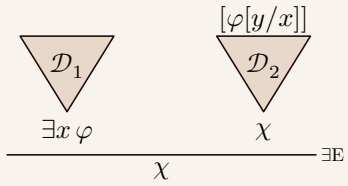
Let d be an element and $s' = s[d/y]$. Then

the formulas of $\mathrm{OH}(\mathcal{D}')$ are true under s'	by relevance,
$\varphi[y/x]$ is true under s'	by the induction hypothesis,
φ is true under $s'[d/x]$	by the substitution lemma.

$s'[d/x]$ differs from $s[d/x]$ at most at y, not free in φ unless it is x: by relevance, φ is true under $s[d/x]$.

6.5 · SOUNDNESS OF NJ

Soundness: (∃E)



y free neither in χ, nor in $\exists x\,\varphi$, nor in the other open hypotheses of $\mathcal{D}_2$. By the induction hypothesis for $\mathcal{D}_1$, some d makes φ true under $s[d/x]$. Let $s' = s[d/y]$:

$\varphi[y/x]$ is true under s'	by relevance and substitution,
the rest of $\mathrm{OH}(\mathcal{D}_2)$ is true under s'	by relevance,
χ is true under s'	by the induction hypothesis,
χ is true under s	by relevance.

6.5 · SOUNDNESS OF NJ

Four rules carried the weight

- Two discharge a supposition, ($\to$I) and ($\vee$E): there the open hypotheses of the whole are fewer than those of the parts.
- Two have a side condition on a variable, ($\forall$I) and ($\exists$E): there the relevance and substitution lemmas were needed.
- The other seven follow from the clause for their connective or quantifier, two of them with the substitution lemma as well.

6.5 · SOUNDNESS OF NJ

Example 6.40: a derivation

The entailment of Example 6.37 has a derivation, the argument of that example written as a tree:

$$\Theta \vdash_{\mathrm{NJ}} \forall x\,(x \neq \mathsf{ceo} \to \mathsf{boss}(x) \neq x)$$

"nobody but the chief executive is her own boss"

In words: let x be anyone, and suppose that x is not the chief executive and that x is her own boss.

- By R_2, the boss of x earns more than x.
- Since the boss of x *is* x, the equality axiom $\mathrm{E}_{\mathsf{Richer}}$ gives that x earns more than x.
- By R_1 that is absurd. Hence x is not her own boss.

6.5 · SOUNDNESS OF NJ

Example 6.40: the piece $\mathcal{D}_1$

$\mathcal{D}_1$ instantiates $\mathrm{E}_{\mathsf{Richer}}$ three times and uses the hypothesis $t = x$, where t is $\mathsf{boss}(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{E}_{\mathsf{Richer}}}{\forall v\,\forall w\,(t = v \to (\mathsf{Richer}(t,w) \to \mathsf{Richer}(v,w)))}\,\forall\mathrm{E}}{\forall w\,(t = x \to (\mathsf{Richer}(t,w) \to \mathsf{Richer}(x,w)))}\,\forall\mathrm{E}}{t = x \to (\mathsf{Richer}(t,x) \to \mathsf{Richer}(x,x))}\,\forall\mathrm{E} \qquad t = x}{\mathsf{Richer}(t,x) \to \mathsf{Richer}(x,x)}\,\to\mathrm{E}$$

6.5 · SOUNDNESS OF NJ

Example 6.40: the pieces $\mathcal{D}_2$ and $\mathcal{E}$

$\mathcal{D}_2$ uses R_2 and the hypothesis $x \neq \mathsf{ceo}$:

$$\dfrac{\dfrac{\mathrm{R}_2}{x \neq \mathsf{ceo} \to \mathsf{Richer}(t, x)}\,\forall\mathrm{E} \qquad x \neq \mathsf{ceo}}{\mathsf{Richer}(t, x)}\,\to\mathrm{E}$$

$\mathcal{E}$ combines the two and meets R_1:

$$\dfrac{\dfrac{\mathrm{R}_1}{\neg\mathsf{Richer}(x, x)}\,\forall\mathrm{E} \qquad \dfrac{\mathcal{D}_1 \qquad\qquad \mathcal{D}_2}{\mathsf{Richer}(t, x) \to \mathsf{Richer}(x, x) \qquad \mathsf{Richer}(t, x)}\,\to\mathrm{E}}{\bot}\,\to\mathrm{E}$$

Open hypotheses of $\mathcal{E}$: R_1, R_2, $\mathrm{E}_{\mathsf{Richer}}$, $x \neq \mathsf{ceo}$ and $t = x$.

Example 6.40: the last piece

The last piece discharges the two suppositions and generalises:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[x \neq \mathsf{ceo}]^1 \quad [t = x]^2 \\ \vdots\ \mathcal{E} \\ \bot\end{array}}{t \neq x}\ {\to}\mathrm{I},\,2}{x \neq \mathsf{ceo} \to t \neq x}\ {\to}\mathrm{I},\,1}{\forall x\,(x \neq \mathsf{ceo} \to \mathsf{boss}(x) \neq x)}\ \forall\mathrm{I}$$

($\forall$I) is legal: once both suppositions are discharged, the only open hypotheses are the sentences R_1, R_2, $\mathrm{E}_{\mathsf{Richer}}$.

6.5 · SOUNDNESS OF NJ

Example 6.40: the tree and the argument

IN THE ARGUMENT OF EXAMPLE 6.37	IN THE TREE
"let d be an element"	the ($\forall$I) at the root
"suppose that"	each ($\rightarrow$I)
"by R_2"	a ($\forall$E) followed by a ($\rightarrow$E)

By soundness the tree is a second proof of the entailment, and one that a program can check.

SECTION 6.6

The Law of the Excluded Middle

Safe, but not enough; one rule with no premisses.

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Theorem 6.41: incompleteness of NJ

Is NJ also complete? It is not.

THEOREM 6.41 · INCOMPLETENESS OF NJ

There are a set of formulas Γ and a formula φ such that $\Gamma \vDash \varphi$ but $\Gamma \nvdash_{\mathrm{NJ}} \varphi$.

"some entailment has no derivation in NJ"

The proof uses a property of NJ *proved in Part 1: a derivation of a disjunction from no hypotheses says which disjunct holds.*

Fact 6.42: disjunction property, from Part 1

FACT 6.42 · DISJUNCTION PROPERTY, FROM PART 1

If $\vdash_{\mathrm{NJ}} \psi \vee \chi$, then $\vdash_{\mathrm{NJ}} \psi$ or $\vdash_{\mathrm{NJ}} \chi$.

"a proof from nothing of "ψ or χ" says which"

Why: every derivation can be straightened into one with no detours, and a derivation with no detours and no open hypotheses ends with an introduction rule. For $\psi \vee \chi$ that is ($\vee$I), and the tree above it derives ψ or derives χ.

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Theorem 6.41: the proof

Take Γ empty, and φ the sentence $p \vee \neg p$ for a propositional letter p.

- *Valid.* In every structure, under every assignment, p is true or false, and when it is false, $\neg p$ is true.
- *No derivation.* If $\vdash_{\mathrm{NJ}} p \vee \neg p$, then by the disjunction property NJ derives p or $\neg p$, and by soundness that disjunct is valid.
- Neither is: in a structure that makes p true, $\neg p$ is false; in one that makes p false, p is false.

So close

$$\vdash_{\mathrm{NJ}} \neg\neg(\varphi \vee \neg\varphi)$$

"NJ proves that excluded middle cannot be refuted"

- $\varphi \vee \neg\varphi$ is the **law of the excluded middle**: every structure obeys it, and NJ lacks it.
- The rules come within one step: no rule leads from $\neg\neg\varphi$ to φ.

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Remark 6.43: two senses of incompleteness

OF A PROOF SYSTEM

Complete if it derives everything that is entailed; incomplete if some entailment has no derivation. The sense of this section.

OF A THEORY

Incomplete when some truth about its subject matter cannot be derived from it. The sense of Chapter 8.

Example 6.44: one negation too many

The finance director could have put his question the other way round: whoever is their own boss is the chief executive.

$$\forall x \, (\mathsf{boss}(x) = x \to x = \mathsf{ceo})$$

"only the chief executive is her own boss"

- True in the same structures as the sentence of Example 6.40, its contrapositive; and Θ entails it.
- Let $\mathfrak{M} \models \Theta$ and d be her own boss. *Either d is the chief executive or not.* If so, nothing to show; if not, the argument of Example 6.37 gives a contradiction.

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Example 6.44: what the rules provide

$$\Theta \vdash_{\mathrm{NJ}} \forall x\,(\mathsf{boss}(x) = x \to x = \mathsf{ceo})\ ?$$

WHAT THE ARGUMENT NEEDS

Either x is the chief executive or not. The rules do not provide it.

WHAT THE RULES PROVIDE

Suppose $\mathsf{boss}(x) = x$ and $x \neq \mathsf{ceo}$; then absurdity, as in Example 6.40; hence $\neg(x \neq \mathsf{ceo})$, which is $\neg\neg(x = \mathsf{ceo})$.

One negation too many. No rule of NJ *makes the last step, and no cleverer derivation finds a way round.*

6.6 · THE LAW OF THE EXCLUDED MIDDLE

The missing law

- The gap in NJ is a principle every structure obeys: of a claim and its denial, one is true, and there is no third option in the middle.
- In a structure it holds by bivalence: the clause for negation makes exactly one of φ and $\neg\varphi$ true.
- In a proof system it becomes a rule with no premisses, for every formula φ:

$$\frac{}{\varphi \vee \neg\varphi}\ \text{LEM}$$

"at any point, you may assert φ or not φ"

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Three more in the gap

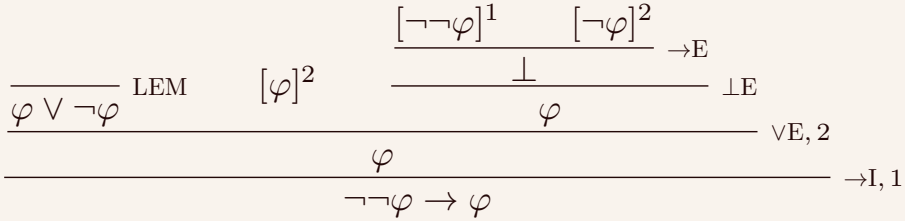
$$\neg\neg\varphi \to \varphi \qquad ((\varphi \to \psi) \to \varphi) \to \varphi \qquad \neg\forall x\, \neg\varphi \to \exists x\, \varphi$$

- *Double negation*: what is not false is true.
- *Peirce's law* says much the same without negation.
- The third: if not everything lacks the property φ, then something has it.

Every instance is valid; but NJ *derives none of* $\neg\neg p \to p$, $((p \to q) \to p) \to p$, $\neg\forall x\, \neg P(x) \to \exists x\, P(x)$.

6.6 · THE LAW OF THE EXCLUDED MIDDLE

With the law, double negation



We write (DN) for this derivation used as a step.

Definition 6.45: NK

The system NK is NJ with the law of the excluded middle added as a rule with no premisses.

The derivations of NK are built by the rules of NJ together with the rule (LEM).

- Every derivation of NJ is one of NK: if $\Gamma \vdash_{\mathrm{NJ}} \varphi$ then $\Gamma \vdash_{\mathrm{NK}} \varphi$.
- J for *intuitionistisch*, K for *klassisch*. The system with the law added is **classical logic**.

Example 6.46: a derivation in NK

$$\Theta \vdash_{\mathrm{NK}} \forall x\,(\mathsf{boss}(x) = x \to x = \mathsf{ceo})$$

"only the chief executive is her own boss"

The piece $\mathcal{E}$ of Example 6.40 derives $\bot$ from $x \neq \mathsf{ceo}$ and $\mathsf{boss}(x) = x$. The new step splits on $x = \mathsf{ceo}$:

$$\dfrac{\dfrac{\dfrac{}{x = \mathsf{ceo} \vee x \neq \mathsf{ceo}}\,\mathrm{LEM} \qquad \dfrac{\dfrac{[x \neq \mathsf{ceo}]^2 \quad [\mathsf{boss}(x) = x]^1}{\mathcal{E}}}{\bot}}{x = \mathsf{ceo}}\,\bot\mathrm{E}}{x = \mathsf{ceo}}\,\vee\mathrm{E},\,2}{\mathsf{boss}(x) = x \to x = \mathsf{ceo}}\,\to\mathrm{I},\,1}{\forall x\,(\mathsf{boss}(x) = x \to x = \mathsf{ceo})}\,\forall\mathrm{I}$$

6.6 · THE LAW OF THE EXCLUDED MIDDLE

Theorem 6.47: soundness of NK

THEOREM 6.47 · SOUNDNESS OF NK

For every set Γ of formulas and every formula φ, if $\Gamma \vdash_{\mathrm{NK}} \varphi$ then $\Gamma \vDash \varphi$.

"the new rule is safe too"

Proof. The soundness proof for NJ goes through with one more case, the rule (LEM): its conclusion $\varphi \vee \neg\varphi$ rests on no open hypotheses, so it must be true in every structure under every assignment. By bivalence it is.

SECTION 6.7

Completeness of NK

Whatever follows has a derivation.

6.7 · COMPLETENESS OF NK

The other half

- Soundness puts derivability inside entailment: whatever NK derives from Γ is true in every model of Γ.
- For NJ the converse failed, and excluded middle was the missing step.
- For NK the converse holds.

6.7 · COMPLETENESS OF NK

Theorem 6.48: completeness

THEOREM 6.48 · COMPLETENESS

Let Σ be a countable signature, Γ a set of Σ-sentences and φ a Σ-sentence. If $\Gamma \vDash \varphi$, then $\Gamma \vdash_{\mathrm{NK}} \varphi$.

"whatever is true in every model of a theory can be derived from it in NK"

Gödel proved it in 1929 for a Hilbert-style system. The proof sketched here is Henkin's (1949).

6.7 · COMPLETENESS OF NK

Turn it round

- The antecedent is the hard side: $\Gamma \vDash \varphi$ speaks of every model of Γ, and gives no hint of a finite derivation.
- The standard move when an antecedent is hard to use: prove the contrapositive.

$$\text{if } \Gamma \nvdash_{\mathrm{NK}} \varphi, \text{ then } \Gamma \nvDash \varphi$$

"a failure to derive yields a counter-model"

The antecedent is now a failure to derive, which seems to carry no information; but the conclusion asks for only one *structure, and a single structure is something we can hope to build.*

6.7 · COMPLETENESS OF NK

One set, two rephrasings

Both sides of the contrapositive are facts about one set, $\Gamma \cup \{\neg\varphi\}$.

THE CONCLUSION

Some structure satisfies Γ and falsifies φ. By the clause for negation: $\Gamma \cup \{\neg\varphi\}$ *has a model.*

THE ANTECEDENT

φ has no derivation from Γ: then $\bot$ has none from $\Gamma \cup \{\neg\varphi\}$. Otherwise ($\to$I) gives $\neg\neg\varphi$ from Γ, and (DN) gives φ.

A set from which $\bot$ cannot be derived has a name.

6.7 · COMPLETENESS OF NK

Definition 6.49: consistency

A consistent set is one from which the rules cannot derive absurdity.

A set Γ of Σ-sentences is **consistent** if $\Gamma \nvdash_{\mathrm{NK}} \bot$, and **inconsistent** otherwise.

- *Consistent* is about derivations; *having a model* is about structures.
- Soundness gives one direction: a set with a model is consistent, since $\bot$ is true in no structure.

6.7 · COMPLETENESS OF NK

Lemma 6.50: two reformulations

LEMMA 6.50 · TWO REFORMULATIONS

Let Γ be a set of sentences and φ a sentence.

(i) $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ has a model.
(ii) $\Gamma \vdash_{\mathrm{NK}} \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is consistent.

(i) by the clause for negation. (ii) by ($\to$I) and (DN) one way, by ($\to$E) the other.

Figure 6.8: from completeness to model existence

completeness: if $\Gamma \vDash \varphi$, then $\Gamma \vdash_{\mathrm{NK}} \varphi$

↓ the contrapositive

if $\Gamma \nvdash_{\mathrm{NK}} \varphi$, then $\Gamma \nvDash \varphi$

↓ reformulation (ii) on the left, (i) on the right

if $\Gamma \cup \{\neg\varphi\}$ is consistent, then it has a model

↓ forget where the set came from

model existence: if Θ is consistent, then Θ has a model

6.7 · COMPLETENESS OF NK

Lemma 6.51: model existence

LEMMA 6.51 · MODEL EXISTENCE

Let Σ be a countable signature. Every consistent set of Σ-sentences has a model.

"a theory that does not contradict itself describes something"

Completeness from the lemma: if $\Gamma \nvdash_{\mathrm{NK}} \varphi$, then

$\Gamma \cup \{\neg\varphi\}$ is consistent	by reformulation (ii),
$\Gamma \cup \{\neg\varphi\}$ has a model	by model existence,
$\Gamma \not\models \varphi$	by reformulation (i).

6.7 · COMPLETENESS OF NK

A theory that behaves like a model

- We are handed a consistent Θ and must produce a structure. The only material in sight is Θ itself.
- So the elements will be the names the signature provides, the **closed terms**, and a relation will hold of some names exactly when Θ says it does. The structure is made of its own names.
- For which theories can that work? The sentences true in a structure where every element has a name behave as the clauses of satisfaction say.

A theory that behaved in the same way would carry all a structure carries. So we make the clauses into a definition.

6.7 · COMPLETENESS OF NK

Definition 6.52: model-like theory

A model-like theory treats the connectives and quantifiers as satisfaction does, with membership in place of truth and closed terms in place of elements.

A set Θ of Σ-sentences is **model-like** if the clauses of Figure 6.9 hold for all Σ-sentences ψ and χ, and, in the two quantifier clauses, for every Σ-formula ψ whose only free variable is x.

A model-like theory is the list of truths of a structure, kept by someone who has the list and nothing else.

Figure 6.9: the clauses of a model-like theory

$$
\begin{array}{lll}
\bot \notin \Theta & & \\
\psi \wedge \chi \in \Theta & \text{iff} & \psi \in \Theta \text{ and } \chi \in \Theta \\
\psi \vee \chi \in \Theta & \text{iff} & \psi \in \Theta \text{ or } \chi \in \Theta \\
\psi \to \chi \in \Theta & \text{iff} & \psi \notin \Theta \text{ or } \chi \in \Theta \\
\forall x\, \psi \in \Theta & \text{iff} & \psi[t/x] \in \Theta \text{ for every closed term } t \\
\exists x\, \psi \in \Theta & \text{iff} & \psi[t/x] \in \Theta \text{ for some closed term } t
\end{array}
$$

6.7 · COMPLETENESS OF NK

Definition 6.53: term structure

The term structure of a set of sentences is made of the closed terms, related as the set says they are.

Let Σ have at least one constant symbol, and Θ be a set of Σ-sentences. Its **term structure** is $\mathfrak{M}_\Theta = (\mathbb{D}_\Theta, I_\Theta)$, where $\mathbb{D}_\Theta$ is the set of closed Σ-terms and

$$[\![c]\!]^{\mathfrak{M}_\Theta} = c, \qquad [\![f]\!]^{\mathfrak{M}_\Theta}(t_1, \dots, t_n) = f(t_1, \dots, t_n),$$

$$[\![R]\!]^{\mathfrak{M}_\Theta} = \{(t_1, \dots, t_n) \in \mathbb{D}_\Theta^n : R(t_1, \dots, t_n) \in \Theta\}.$$

Every closed term names itself. The difficulty of an element without a name cannot arise: the elements are *the names.*

6.7 · COMPLETENESS OF NK

Lemma 6.54: truth lemma

LEMMA 6.54 · TRUTH LEMMA

Let Σ have at least one constant symbol and Θ be a model-like set of Σ-sentences. For every Σ-sentence ψ, $\mathfrak{M}_\Theta \models \psi$ iff $\psi \in \Theta$.

"in the structure made of names, a sentence is true exactly when Θ says it is"

- *Atoms*: the definition of $[\![R]\!]^{\mathfrak{M}_\Theta}$, since every closed term names itself. $\bot$: neither side holds.
- *Each other case*: a clause of satisfaction meets the matching clause of Figure 6.9; for the quantifiers, every element is a name.

6.7 · COMPLETENESS OF NK

Corollary 6.55: a model-like theory has a model

COROLLARY 6.55 · A MODEL-LIKE THEORY HAS A MODEL

Over a signature with at least one constant symbol, every model-like set of sentences Θ has a model, its term structure $\mathfrak{M}_\Theta$.

So the task is now to turn a given consistent theory into a model-like one.

6.7 · COMPLETENESS OF NK

What a consistent theory already has

- Call Θ **closed under derivation** if every sentence that NK derives from Θ is in Θ.
- Closing a consistent theory costs nothing: add everything it derives, and it stays consistent, since whatever the larger set derives, the original already did.

A consistent theory closed under derivation already satisfies half of Figure 6.9, one rule for each half-clause.

6.7 · COMPLETENESS OF NK

Lemma 6.56: what closure gives

Let Θ be consistent and closed under derivation. For sentences ψ, χ, and ψ with only x free:

PART	WHAT Θ HAS	BY
(i)	$\bot \notin \Theta$	consistency
(ii)	$\psi \wedge \chi \in \Theta$ iff $\psi \in \Theta$ and $\chi \in \Theta$	($\wedge$E), ($\wedge$I)
(iii)	if $\psi \in \Theta$ or $\chi \in \Theta$, then $\psi \vee \chi \in \Theta$	($\vee$I)
(iv)	if $\psi \to \chi, \psi \in \Theta$, then $\chi \in \Theta$; if $\chi \in \Theta$, then $\psi \to \chi \in \Theta$	($\to$E); ($\to$I)
(v)	if $\forall x\, \psi \in \Theta$, then $\psi[t/x] \in \Theta$ for every closed t	($\forall$E)
(vi)	if $\psi[t/x] \in \Theta$ for some closed t, then $\exists x\, \psi \in \Theta$	($\exists$I)

Each rule derives the sentence the part puts in Θ, and closure puts it there.

6.7 · COMPLETENESS OF NK

Four halves missing: which disjunct, which witness

CLAUSE	THE MISSING HALF
$\vee$	from $\psi \vee \chi \in \Theta$ to $\psi \in \Theta$ or $\chi \in \Theta$
$\rightarrow$	from $\psi \notin \Theta$ to $\psi \rightarrow \chi \in \Theta$
$\exists$	from $\exists x\, \psi \in \Theta$ to some instance $\psi[t/x] \in \Theta$
$\forall$	from every instance $\psi[t/x] \in \Theta$ to $\forall x\, \psi \in \Theta$

Two pieces of information a theory can lack and a structure cannot: which disjunct, *and* which witness.

A consistent theory can contain $p \vee q$ without p or q, and $\exists x\, P(x)$ without any $P(t)$. Lindenbaum supplies the first two halves; Henkin the last two.

6.7 · COMPLETENESS OF NK

Definition 6.57: maximal consistency

A maximal consistent set is a consistent set that takes a side on every sentence.

A set Δ of Σ-sentences is **maximal consistent** if

- it is consistent, and
- for every Σ-sentence ψ, either $\psi \in \Delta$ or $\neg\psi \in \Delta$.

A structure makes every sentence true or false; a consistent set may be silent about many.

6.7 · COMPLETENESS OF NK

Lemma 6.58: Lindenbaum

LEMMA 6.58 · LINDENBAUM

Let Σ be countable. Every consistent set Θ of Σ-sentences is contained in a maximal consistent set of Σ-sentences, which we write $\mathrm{L}(\Theta)$.

"a set that does not contradict itself can be enlarged until it takes a side on every question, and still not contradict itself"

6.7 · COMPLETENESS OF NK

Lindenbaum's construction

List all sentences as $\psi_0, \psi_1, \psi_2, \ldots$ (finite strings over a countable alphabet), and decide each in turn: $\Delta_0 = \Theta$ and

$$\Delta_{n+1} := \begin{cases} \Delta_n \cup \{\psi_n\} & \text{if that set is consistent,} \\ \Delta_n \cup \{\neg\psi_n\} & \text{otherwise.} \end{cases}$$

Let $\mathrm{L}(\Theta) = \bigcup_n \Delta_n$.

- Every sentence or its negation is in $\mathrm{L}(\Theta)$; each Δ_n stays consistent, by induction on n.
- $\mathrm{L}(\Theta)$ is consistent: a derivation of $\bot$ uses finitely many hypotheses, all in some Δ_n.

Not an algorithm: it asks whether a set is consistent, and needs only that the question has an answer.

6.7 · COMPLETENESS OF NK

Lemma 6.59: maximal consistent sets

A set that takes a side on every sentence supplies the two propositional clauses.

LEMMA 6.59 · MAXIMAL CONSISTENT SETS

Let Δ be maximal consistent. Then

(i) Δ is closed under derivation;
(ii) if $\psi \vee \chi \in \Delta$, then $\psi \in \Delta$ or $\chi \in \Delta$;
(iii) if $\psi \notin \Delta$, then $\psi \to \chi \in \Delta$.

"a maximal consistent set satisfies every clause of Figure 6.9 except, perhaps, the two for the quantifiers"

In each part, suppose the conclusion fails and derive $\bot$ from Δ, against consistency; where $\psi \notin \Delta$, the side taken is $\neg\psi \in \Delta$.

6.7 · COMPLETENESS OF NK

Naming the witnesses

- The quantifier clauses can still fail in a maximal consistent set: it may contain $\exists x\, \psi$ and the negation of every instance $\psi[t/x]$ (Example 6.68 shows a case).
- The remedy is to add names: a new constant symbol for every existential sentence, and a sentence $\exists x\, \psi \to \psi[c/x]$ that makes it the witness.

LEMMA 6.60 · GENERALISATION ON A NEW CONSTANT

If $\Gamma \vdash_{\mathrm{NK}} \psi[c/x]$, where the constant c occurs neither in Γ nor in ψ, then $\Gamma \vdash_{\mathrm{NK}} \forall x\, \psi$.

6.7 · COMPLETENESS OF NK

Lemma 6.61: Henkin

- Add new constants $c_0, c_1, c_2, \ldots$ to Σ, and list the existential sentences of the bigger signature as $\exists x_n\, \psi_n$.
- For each n let θ_n be $\exists x_n\, \psi_n \to \psi_n[d_n/x_n]$, where d_n is a new constant used in neither ψ_n nor any earlier θ_k.

LEMMA 6.61 · HENKIN

Let Σ be countable and Θ a consistent set of Σ-sentences. Then $\mathrm{H}(\Theta) := \Theta \cup \{\theta_0, \theta_1, \ldots\}$ is a consistent set of Σ^+-sentences.

"every existential sentence may be given a named witness, provided each name is new"

Remark 6.62: why the witnesses must be new

$$\{\exists x\, P(x),\ \neg P(c)\}$$

"something is P, but not the thing c names"

- Consistent: any structure with an element of P that c does not name satisfies it.
- Had the witness been the old constant c, the set would derive $P(c)$ and $\neg P(c)$. A name already in use carries information.
- Nor may two existential sentences share a new name (Exercise 6.17).

Lemma 6.63: the quantifier clauses

LEMMA 6.63 · THE QUANTIFIER CLAUSES

Let Δ be a maximal consistent set of Σ^+-sentences that contains $\mathrm{H}(\Theta)$. For every Σ^+-formula ψ whose only free variable is x:

(i) if $\exists x\,\psi \in \Delta$, then $\psi[t/x] \in \Delta$ for some closed term t;
(ii) if $\psi[t/x] \in \Delta$ for every closed term t, then $\forall x\,\psi \in \Delta$.

(i): $\exists x\,\psi$ is some $\exists x_n\,\psi_n$, and $\theta_n \in \Delta$ gives the instance by ($\to$E). (ii): otherwise $\neg\forall x\,\psi \in \Delta$, hence $\exists x\,\neg\psi$, and (i) gives an instance $\neg\psi[t/x]$: a contradiction.

6.7 · COMPLETENESS OF NK

Figure 6.10: putting it together

THE SET	WHAT IT IS	BY
Θ	consistent, over Σ	given
$\mathrm{H}(\Theta)$	consistent, over Σ^{+}, with named witnesses	the Henkin lemma
$\Delta = \mathrm{L}(\mathrm{H}(\Theta))$	maximal consistent, hence model-like	Lindenbaum; the three lemmas
$\mathfrak{M}_{\Delta}$	true exactly when a member of Δ	the truth lemma
$\mathfrak{M}_{\Delta}{\restriction}\Sigma$	a model of Θ	forget the new constants

Henkin names the witnesses, Lindenbaum decides every sentence, and the three lemmas say that the result is model-like.

6.7 · COMPLETENESS OF NK

A structure made of words

- Asked to show that a sentence true in every structure has a proof, we turned the sentences themselves into a structure made of words.
- Nothing in the definition of a structure forbids it: a domain may be any non-empty set, and a set of strings is a set like any other.

Every consistent set has a model; so whatever follows has a derivation in NK.

6.7 · COMPLETENESS OF NK

Remark 6.64: where excluded middle was used

REFORMULATION (II)

From $\neg\neg\varphi$ to φ, by (DN): a failure to derive φ makes $\Gamma \cup \{\neg\varphi\}$ consistent.

THE QUANTIFIER CLAUSES

From $\neg\forall x\,\psi$ to $\exists x\,\neg\psi$: a relative of the quantifier law that NJ could not derive.

Nowhere else. At the first, the argument already breaks down for NJ*: it does not derive $p \vee \neg p$, yet $\{\neg(p \vee \neg p)\}$ is inconsistent.*

6.7 · COMPLETENESS OF NK

Remark 6.65: compactness; Löwenheim–Skolem

COMPACTNESS

If every finite subset of Γ has a model, so does Γ: a derivation uses only finitely many hypotheses.

LÖWENHEIM–SKOLEM

The structure built is made of terms, countably many: a set with a model has one whose domain can be listed as $0, 1, 2, \ldots$

Both say that first-order sentences discriminate less than one might hope.

Remark 6.66: normal structures

- The tables a system produces interpret $=$ as identity. The term structure need not: Δ may contain $c = d$ for different constants.
- If Θ contains the equality axioms, the defect can be repaired: lump together the terms that Δ says are equal. The classes form a normal structure where the same sentences are true.
- So if φ is true in every normal model of Θ, then $\Theta \vdash_{\mathrm{NK}} \varphi$.

6.7 · COMPLETENESS OF NK

Example 6.67: Lindenbaum's construction on three letters

$\Gamma = \{p \vee q,\ q \to r\}$ and $\varphi = r$. Start from $\Delta_0 = \{p \vee q,\ q \to r,\ \neg r\}$; the list begins p, q, r.

SENTENCE	ADDED WITH IT, CONSISTENT?	GOES IN
p	yes	p
q	no: q and $q \to r$ give r, against $\neg r$	$\neg q$
r	no: it meets $\neg r$	$\neg r$

A structure that makes p true and q, r false, as the decisions say, makes $p \vee q$ and $q \to r$ true and r false: the entailment fails.

Example 6.68: an engineer with no name

- In $\mathfrak{C}$ every closed term names Thorne: ceo is Thorne, and Thorne is her own boss.
- The set of sentences true in $\mathfrak{C}$ is maximal consistent, and contains $\exists x\, \mathsf{Eng}(x)$ and also $\neg\mathsf{Eng}(t)$ for every closed term t.
- So the clause for $\exists$ of Figure 6.9 fails: the set is not model-like, and its term structure would contain no engineer.

Example 6.69: the finance director's two answers

CASE: ONE RELATION

Take Γ to be the theory Θ of the personnel system. By soundness and completeness, "true in every model of Θ" and "derivable from Θ in NK" are one relation.

The finance director's question has two answers, and they are the same answer.

SECTION 6.8

Classical and Intuitionistic Logic

Two logics, one language.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Two logics for one language

NJ

The logic of Part 1, whose proofs are programs.

NK

Complete for the structures of this chapter.

- The gain: entailment, over every structure, becomes derivability, a finite certificate. An argument by cases on any claim is available, whether or not anyone can tell which case holds.
- The cost: adding a rule can only add derivations, and the new ones cost NK two properties of NJ: the disjunction property (Fact 6.42) and the existence property.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Fact 6.70: existence property, from Part 1

FACT 6.70 · EXISTENCE PROPERTY, FROM PART 1

If $\vdash_{\mathrm{NJ}} \exists x\, \varphi$, then $\vdash_{\mathrm{NJ}} \varphi[t/x]$ for some term t.

"a proof from nothing of "there is an x such that φ" names a term for x"

The companion of the disjunction property (Fact 6.42), and proved the same way: a straightened derivation ends with ($\exists I$).

Remark 6.71: the drinker principle

One one-place relation symbol Dr; the domain is the customers in a pub (at least one, since domains are not empty); $\mathsf{Dr}(x)$ reads "x is drinking".

$$\exists x\,(\mathsf{Dr}(x) \to \forall y\,\mathsf{Dr}(y))$$

"there is someone in the pub such that, if they are drinking, everyone is"

It sounds like a claim about a remarkable person, and so it is often suspected of being false.

The name is Smullyan's (1978), who called it the drinking principle.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Remark 6.71: why it is valid

Either everyone is drinking or someone is not.

EVERYONE IS DRINKING

Pick anyone: the consequent is true, so the implication is. This case uses the non-empty domain.

SOMEONE IS NOT

Pick that person: the antecedent is false, so the implication is true.

There is no remarkable person, only a choice made differently in the two cases.

Remark 6.71: NJ refuses it

- If NJ derived it, by the existence property (Fact 6.70) it would derive $\mathsf{Dr}(t) \to \forall y\, \mathsf{Dr}(y)$ for some term t: here a variable, say z.
- But that is false in a pub of two under an assignment sending z to the one who is drinking. By soundness, no derivation.
- Asked for *someone*, the rules of NJ insist on being told who. The argument never says: "this one if everyone is drinking, that one if not".

In a pub of five, no defect: go and look. Over all the counting numbers, with a property nobody can check, the same argument proves a number exists without saying how to find it.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Example 6.72: the drinker in NK, someone is not

Write δ for the drinker principle. If someone is not drinking, call them z: the antecedent fails for z. The derivation $\mathcal{F}_1$:

$$\dfrac{[\exists y\, \neg \mathsf{Dr}(y)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg \mathsf{Dr}(z)]^2 \qquad [\mathsf{Dr}(z)]^3}{\bot}\,\to\!\mathrm{E}}{\forall y\, \mathsf{Dr}(y)}\,\bot\mathrm{E}}{\mathsf{Dr}(z) \to \forall y\, \mathsf{Dr}(y)}\,\to\!\mathrm{I},3}{\delta}\,\exists\mathrm{I}}{\delta}\,\exists\mathrm{E},2$$

Example 6.72: nobody is not

If nobody is not drinking, everyone is, and anyone, say x, is the witness. The derivation $\mathcal{F}_2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists y\,\neg \mathsf{Dr}(y)]^1 \qquad \dfrac{[\neg \mathsf{Dr}(y)]^4}{\exists y\,\neg \mathsf{Dr}(y)}\,\exists\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg \mathsf{Dr}(y)}\,{\to}\mathrm{I},4}{\mathsf{Dr}(y)}\,\mathrm{DN}}{\forall y\,\mathsf{Dr}(y)}\,\forall\mathrm{I}}{\mathsf{Dr}(x) \to \forall y\,\mathsf{Dr}(y)}\,{\to}\mathrm{I}}{\delta}\,\exists\mathrm{I}$$

The ($\to$I) step discharges nothing. The last ($\exists$I) uses a variable as the witness: there the non-empty domain is paid for.

Example 6.72: joining the cases

One step of (∨E) on an instance of (LEM):

$$\dfrac{\dfrac{}{\exists y\,\neg \mathsf{Dr}(y) \vee \neg\exists y\,\neg \mathsf{Dr}(y)}\ \text{LEM} \qquad \begin{array}{c}[\exists y\,\neg \mathsf{Dr}(y)]^1 \\ \mathcal{F}_1 \\ \delta\end{array} \qquad \begin{array}{c}[\neg\exists y\,\neg \mathsf{Dr}(y)]^1 \\ \mathcal{F}_2 \\ \delta\end{array}}{\delta}\ \vee\text{E}, 1$$

Which case supplies the conclusion depends on which side of the (LEM) is true, and the derivation does not say.

Proposition 6.73: loss of the disjunction and existence properties

PROPOSITION 6.73 · LOSS OF THE DISJUNCTION AND EXISTENCE PROPERTIES

The system NK has neither the disjunction property nor the existence property.

- *Disjunction.* NK derives $p \vee \neg p$ in one step, and neither disjunct: neither is valid, and NK is sound.
- *Existence.* NK derives the drinker principle. In a structure with two elements, Dr true of a only, and every variable sent to a, each $\mathsf{Dr}(t) \to \forall y\, \mathsf{Dr}(y)$ is false: by soundness NK derives no such formula.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

It no longer says which

IN NJ

A proof of $\exists x\, \varphi$ from no hypotheses can be straightened into one that ends by exhibiting a term t. The rules were designed so.

IN NK

A proof can conclude $\exists x\, \varphi$ by showing that both answers to some question yield an x, without saying which answer is true.

It certifies that something exists, but no longer shows which thing.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

When the price is felt

- Over a finite structure the price is nothing: whether φ or $\neg\varphi$ holds is settled by running through the domain.
- It is felt only over domains nobody can run through, such as the counting numbers.
- Whether it is worth paying depends on what makes a sentence true: a question the mathematics can sharpen but not settle.

The intuitionists

- Those who refuse the law take the loss of witnesses to be the point. The law was abstracted from finite collections, where going through the list settles every question, and carried over unexamined to infinite ones.
- Goldbach's conjecture, every even number greater than 2 is the sum of two primes, is a sentence about the counting numbers, true or false in the structure they form.
- To assert "Goldbach or not Goldbach" is to claim the question settled, or settleable, when nobody knows how. For them, "true" can only mean "constructed".

Brouwer (1907, 1908). Goldbach's conjecture has been checked by computer up to 4×10^{18}, which settles nothing.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Proof conditions

Their logic gives the connectives meaning by what counts as a *proof* of a sentence, rather than by truth conditions:

A PROOF OF	IS
$\varphi \wedge \psi$	a proof of each conjunct
$\varphi \to \psi$	a method turning every proof of φ into a proof of ψ
$\varphi \vee \psi$	a proof of one disjunct, *together with which one*
$\exists x\, \varphi$	a particular x, with a proof of φ for it

On this reading the disjunction and existence properties are not accidents of the rules: they are what the connectives mean.

Kolmogorov (1925, 1932), Heyting (1930); Gentzen's NJ (1934) is this logic.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

A semantics of its own; equally safe

STRUCTURES THAT GROW

Intuitionistic logic has a semantics of its own: structures that grow in stages, modelling a mathematician whose knowledge grows. NJ is sound and complete for them.

Kripke (1965).

NO SAFER, NO RISKIER

Every theorem of classical arithmetic translates, with $\neg\neg$ in the right places, into one of intuitionistic arithmetic: if that is consistent, so is classical arithmetic.

Gödel (1933); Gentzen, independently.

Each logic has a semantics it fits exactly. What separates them is meaning.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Meaning and use

- The strongest modern case is about language: meaning is fixed by use, and a grasp of meaning must show in what a speaker can do, above all in recognising a proof.
- Some truth conditions could obtain without anyone being able to recognise it, and no speaker could display a grasp of those. So: proof conditions, and intuitionistic logic.
- Realists reply that we understand more than we can verify. And much of analysis can be done constructively without the philosophy; classical mathematics reads it as mathematics that happens to supply witnesses.

Dummett (1975, 2000); Button and Walsh (2018) on the current debate; Bishop (1967).

Proofs as programs, again

- A proof of a disjunction is a value with a tag. A proof of excluded middle would be a program that, given any claim at all, returns the right tag, and there is no such program.
- So the proof assistants used to verify software, such as Lean and Rocq, are built on intuitionistic type theories, and let a user add the law as an axiom (in Lean, through the axiom of choice).

Griffin (1990): classical proofs are programs too, if programs may jump; the type of `call/cc` is Peirce's law. What such a program returns as a witness may later be taken back.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Two logics for one language, side by side

	NJ (INTUITIONISTIC)	NK (CLASSICAL)
rules	natural deduction, as in Part 1	the same, plus excluded middle
meaning of a connective	proof conditions	truth conditions
semantics it fits exactly	structures that grow in stages	this chapter's structures
complete for these structures	no	yes
disjunction, existence properties	yes	no
proofs as programs	yes	only with jumps

Figure 6.11: two logics for one language.

6.8 · CLASSICAL AND INTUITIONISTIC LOGIC

Classical, by decision

The rest of this part of the course is classical, and that is a decision rather than a default. Its question is what holds in *every* structure, a question about truth, and the theorem that answers it, soundness and completeness together, is a theorem about NK.

We now work in a different logic from the one in which proofs are programs, and we have paid for completeness with the witnesses.

CHAPTER 6

History

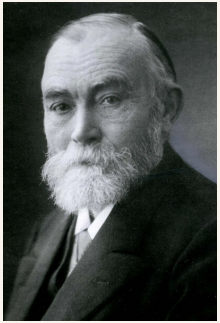
What mathematics may assume.

HISTORY

Frege, 1879

- The language of this chapter is, in modern dress, the one Frege introduced in his *Begriffsschrift*: the first formal language with quantifiers.
- It came with the first system of proof precise enough that checking a proof needs only a comparison of shapes. Every proof checker since descends from it.

Frege's notation was a branching diagram on the page, and almost nobody adopted it; the notation that survived descends from Peirce's.



Gottlob Frege, 1848–1925

HISTORY

Logicism, and a letter

- The nineteenth century rebuilt the calculus on the real numbers, and the reals on sets of rationals. Frege set out to finish the job: arithmetic itself is logic. This is **logicism**, carried out in his *Grundgesetze* (1893).
- June 1902, the second volume at the printer: a letter from Russell. One of Frege's basic laws allows a set of all sets that are not members of themselves, a member of itself exactly when it is not.

> "Hardly anything less welcome can happen to a scientific writer than to have one of the foundations of his building shaken after the work is finished."
>
> Frege, afterword to the second volume.

HISTORY

Three answers

Russell's paradox was one of several, and together they raised the question of what makes mathematics safe.

RUSSELL

The theory of types: the ancestor of the type systems of programming languages.

HILBERT

Keep classical mathematics whole, and prove it safe.

BROUWER

Reject part of it, beginning with the law of the excluded middle.

HISTORY

Hilbert's programme

- Treat proofs as finite strings of symbols, and prove them consistent by methods even Brouwer could not object to. Its questions are our scoreboard's rows.
- 1926: no one shall drive us out of the paradise that Cantor created.
- 1927: taking excluded middle from the mathematician is taking the telescope from the astronomer, or his fists from the boxer.



David Hilbert, 1862–1943

HISTORY

Gödel, 1929 and 1931

- Gödel answered one of Hilbert's questions in his doctoral thesis of 1929, at twenty-three: the completeness theorem of this chapter, for a Hilbert-style system.
- Two years later he showed that the consistency proof Hilbert wanted cannot be given. The last chapter of this part explains why.

HISTORY

Truth before anyone looks

- The definition of satisfaction fits Frege's view of truth: the things are there, determinate, whether or not anyone is looking, and a sentence has its truth value before anyone finds out which.
- Bivalence is no discovery about this picture. It was built into the definition when the clause for negation was written.

Frege, "The Thought" (1918).

HISTORY

Philo's conditional

- The clause for the conditional is older than the rest of the definition. Philo of Megara, around 300 BC, held that a conditional is false only when it leads from a truth to a falsehood.
- His teacher Diodorus Cronus asked that it never could. The dispute is known through Sextus Empiricus.
- Frege's conditional is Philo's, and Russell called it **material implication**.

HISTORY

The paradoxes of material implication

$$\psi \to (\varphi \to \psi) \qquad \neg\varphi \to (\varphi \to \psi) \qquad (\varphi \to \psi) \vee (\psi \to \varphi)$$

"a truth is implied by anything; a falsehood implies anything; of any two sentences, one implies the other"

- All three are valid, though "implies" in English seems to promise a connection.
- Lewis (1912) took them for a defect and proposed *strict* implication; the relevance logics of the 1950s to 1970s asked that the antecedent be used in reaching the consequent.
- Defenders reply that the paradoxes mix up what a conditional says with what makes it worth saying; the conditional of a law is the material one.

Anderson and Belnap (1975); Read (1988); Williamson (2020).

HISTORY

Tarski and Carnap

- Tarski gave the picture a precise form: his definition of truth (1933) is our definition of satisfaction, and his definition of consequence (Paris, 1935) is our definition of entailment.
- It answered Carnap, who had tried to define consequence by the shapes of sentences alone. Tarski used Gödel's incompleteness theorem to show that no finitely many rules of inference can do the job.
- Carnap became the most energetic convert to semantics; others in the Vienna Circle suspected that talk of truth was metaphysics returning by the back door.

HISTORY

Exactly right

- With these definitions Gödel's theorem can be read as showing that, on this picture, the classical system is exactly right: excluded middle is a law of truth, and a system without it fails to see some truths.
- Gentzen's NK of 1934 gave the proofs the form used here.
- The proof of completeness is Henkin's of 1949. He found it by accident, as a graduate student at Princeton, trying to prove something else; it is now the proof in every textbook.

HISTORY

Classical logic and its semantics: who did what

WHO	WHEN	WHAT
Frege	1879	quantifiers; proofs checked by their shape
Frege; Russell	1893–1902	logicism; the paradox in its laws
Hilbert	1920s	a programme to prove mathematics safe
Gödel	1929	completeness
Gödel	1931	incompleteness
Tarski	1933; 1936	definitions of truth and of consequence
Gentzen	1934	NJ and NK
Henkin	1949	the proof of completeness used here

Figure 6.12: classical logic and its semantics: who did what.

SUMMARY

Chapter 6

- A system is a **structure**: a non-empty set of things, with a meaning for each symbol of a signature. A sentence is **true** or false in it by a definition with one clause for each kind of formula; its quantifier clauses loop over the domain, so on a finite structure the definition is an algorithm.
- A theory **entails** a sentence when the sentence is true in every model of the theory. The derivations of NJ are finite certificates, and **sound**; they miss some entailments, and the whole gap is one law, excluded middle.
- NK adds it, and is sound and **complete**: a theory entails a sentence exactly when the sentence has a derivation from it. The price: a proof of an existential sentence no longer says what exists.

SCOREBOARD

Hilbert's programme, after this chapter

QUESTION	STATUS, FOR THE LOGIC OF THIS CHAPTER
consistency	**holds**: NK derives no $\bot$, because $\bot$ is true in no structure and NK is sound
completeness	**holds**: NK derives every sentence true in every model of its hypotheses; NJ does not
decision procedure	**open**: not yet asked. Truth in one finite structure can be computed; truth in *every* structure is another matter

The three questions are Hilbert's, asked of each logic in turn.

LOOKING AHEAD

Chapter 7

NK derives everything a theory entails, but finding an NK derivation of a hard fact is a separate problem: an instance of excluded middle for every formula one could imagine, an unbounded choice of terms for each quantifier, and no hint of which choice leads anywhere.

The completeness proof contains an idea we have not used. It built a counter-model out of a failure to derive: if there is no derivation, there is a structure in which the claim fails.

THE QUESTION

Instead of hunting for the proof, could we hunt for the world?

READING

Sources

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- Button and Walsh (2018); Dummett (2000). What model theory says about what our words mean; the intuitionist case, by its best advocate in English.
- Gentzen (1934); Mancosu (1998); van Dalen (1990). NJ and NK, still readable; the 1920s debate in English; the war of the frogs and the mice.

Thank you.