Introduction to Proof-theoretic Semantics

Lecture 1: General Proof Theory

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Overview

1. What is Logic?
2. Proof Theory
3. Inferentialism
4. Logical Inferentialism

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Figure: Gottlob Frege

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In its modern version, it begins with Gottlob Frege as a means to understand the nature of mathematics.

He introduces a symbolic notation to represent thought and the structure of reasoning.

Figure: Gottlob Frege

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For example, consider the statement $2 + 2 = 4$. In the traditional view, this statement is correct because $2 + 2$ is just another way of saying 4.

While appealing, and doubtless useful, there are some problems with this view. Surely, $2 + 2$ is distinct from 4 because we have to ‘do’ something — namely addition — to get 4 out.

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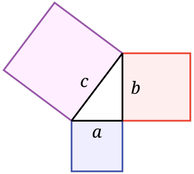

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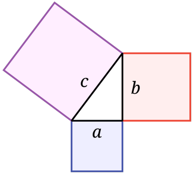

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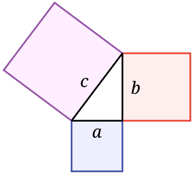

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Question: Can we give a general theory of 'proof'?

The area of logic that concerns this study is known as *proof theory*.

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What they have in common is that proofs are objects constructed through a series of rules over propositions (possibly with additional data).

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Definition (Natural Deduction System NJ.)

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge_{\mathsf{I}} \qquad \frac{\varphi \wedge \psi}{\varphi}\wedge_{\mathsf{E1}} \qquad \frac{\varphi \wedge \psi}{\psi}\wedge_{\mathsf{E2}} \qquad \frac{\begin{matrix}[\varphi]\\ \psi\end{matrix}}{\varphi \to \psi}\to_{\mathsf{I}} \qquad \frac{\bot}{\varphi}\ \mathsf{EFQ}$$

$$\frac{\varphi}{\varphi \vee \psi}\vee_{\mathsf{I1}} \qquad \frac{\psi}{\varphi \vee \psi}\vee_{\mathsf{I2}} \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi]\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \chi\end{matrix}}{\chi}\vee_{\mathsf{E}} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\to_{\mathsf{E}}$$

Example: Disjunctive Syllogism

Consider the following very simple argument:

1. Premiss 1: Either I left my keys at home (H) or I brought them with me (M)
2. Premiss 2: I did not leave my keys at home ($\neg H$)
3. Conclusion: Therefore, I have them with me (M)

It is a proof of $(H \vee M) \wedge \neg H \to M$ and can be represented in NJ as follows:

Examples (An NJ-Proof)

$$
\dfrac{\dfrac{\dfrac{[(H \vee M) \wedge \neg H]^2}{H \vee M} \quad \dfrac{\dfrac{[H]^1 \quad \dfrac{[(H \vee M) \wedge \neg H]^2}{\neg H}}{\bot}}{M}\,\text{EFQ} \quad [M]^1}{M}\,\vee_E^{[1]}}{(H \vee M) \wedge \neg H \to M}\,\to_I^{[2]}
$$

Example: NJ-derivation of $\neg\neg(\varphi \vee \neg\varphi)$

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$$
\dfrac{\dfrac{[(\varphi \vee (\varphi \to \bot)) \to \bot]^1 \qquad \dfrac{\dfrac{\dfrac{[(\varphi \vee (\varphi \to \bot)) \to \bot]^1 \qquad \dfrac{[\varphi]^2}{\varphi \vee (\varphi \to \bot)}\,\vee_{I1}}{\bot}}{\varphi \to \bot}\,\to_I^{[2]}}{\varphi \vee (\varphi \to \bot)}\,\vee_{I2}}{\bot}\,\to_E}{(\varphi \vee (\varphi \to \bot)) \to \bot \to \bot}\,\to_I^{[1]}
$$

Relevant rules:

$$
\dfrac{\begin{matrix}[\varphi]\\ \psi\end{matrix}}{\varphi \to \psi}\,\to_I
$$

$$
\dfrac{\varphi \qquad \varphi \to \psi}{\psi}\,\to_E
$$

$$
\dfrac{\varphi_i}{\varphi_1 \vee \varphi_2}\,\vee_{Ii}
$$

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Both are pursued today and inform each other. Proof theory also comprises a series of subfields such as ordinal analysis, structural proof theory, automated reasoning, and *proof-theoretic semantics*.

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Of course, the category of 'vixen' is expected to be the intersection of the categories of 'fox' and 'female' but that is not explicitly part of the definition of the statement regarding Tammy.

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Expressed as rules:

$$\frac{\text{Tammy is a fox} \qquad \text{Tammy is female}}{\text{Tammy is a vixen}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}}$$

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Furthermore, as a closure condition, what follows from being a parent is what follows from either being a mother or a father:

$$\frac{\text{X is a parent} \qquad \begin{array}{c}[\text{X is a mother}]\\ p\end{array} \qquad \begin{array}{c}[\text{X is a father}]\\ p\end{array}}{p}$$

Example: Parenthood and Childhood

Example

If Tammy is a mother or a father, then she's a mother or a father of someone — she has a child:

$$\frac{\text{X is a mother}}{\text{X has a child}} \qquad \frac{\text{X is a father}}{\text{X has a child}}$$

Clearly, this should then also follow from the fact that she is a parent:

$$\frac{\text{X is a parent} \qquad \begin{array}{c}[\text{X is a mother}]\\ \text{X has a child}\end{array} \qquad \begin{array}{c}[\text{X is a father}]\\ \text{X has a child}\end{array}}{\text{X has a child}}$$

Importantly, from the assumption that 'X is a parent' alone we know 'X has a child' without needing to assume either that 'X is a mother' or 'X is a father' — these hypotheses used in the reasoning have been *discharged*.

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- **Frege** in *Begriffsschrift* (1879) — though critically changes emphasis in *Foundations of Arithmetic* (1884)
- **Wittgenstein** in *Philosophical Investigations* (1953) — argued that meaning is use: the significance of a word lies in how it functions in language-games

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- **Dummett** in *The Logical Basis of Metaphysics* — attempts a justification of intuitionistic logic in terms of anti-realism
- **Brandom** in *Making it Explicit* (1994) — a theory of meaning in language. This is the first instance of the name 'inferentialism'.

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But we want to stay in the comparatively small theatre of **formal logic** where it connects to various other ideas including, for example, constructivism.

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Idea!

The meaning of conjunction *is* this kind of inferential structure.

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He doesn't mean that the eliminations are derived from the introductions (they're not), only that they're justified by them.

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Scope Warning!

These lectures concern some specific ideas in proof-theoretic semantics, but are not exhaustive. Many other fields fall into this subject and may not be inferentialist in spirit at all — for example, the BHK-interpretation of intuitionism is a P-tS broadly conceived.

‘Proof-theoretic Semantics’

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Figure: Peter Schroeder-Heister

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The point is not to restrict ‘semantics’ to truth-conditional approaches alone — after all, ‘semantics’ is the standard term for investigations dealing with the meaning of linguistic expressions.

Figure: Peter Schroeder-Heister

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Such inferential relationships are called *atomic rules*. A collection of such atomic rules $\mathscr{S}$ is an *atomic system*. Given an atomic system $\mathscr{S}$ and a proposition p, we may write $\vdash_{\mathscr{S}} p$ to denote that p can be derived from the rules in $\mathscr{S}$.

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From this set of beliefs, Aristotle is committed to the position that *Socrates is mortal* — that is, $\vdash_{\mathscr{A}} M(s)$, where $M(s)$ denotes 'Socrates is mortal'.