Introduction to Proof-theoretic Semantics

Lecture 3: Base-extension Semantics

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- $\mathfrak{B}_2$ — contains atomic systems with 'second-level' rules,

$$\frac{}{c} \qquad \frac{p_1 \quad \dots \quad p_n}{c} \qquad \frac{[P_1] \qquad\qquad [P_n]}{p_1 \quad \dots \quad p_n \quad c}$$

Other choices may(?) be interesting.

Conceptual Blueprint

In *Validity Concepts in Proof-theoretic Semantics* (2006), Schroeder-Heister outlines three guiding principles for defining the validity of proofs:

- Determine a class of 'canonical' proofs that are *a priori* valid.
- Completed† non-canonical proofs are valid if they contain‡ a canonical proof.
- Uncompleted proofs are valid if any way of closing them produces a valid proof.

Blueprint

You can make this precise in various ways and thereby get different notions of validity.

Proof-theoretic Validity

Altogether this delivers the following definition (with respect to a basis $\mathfrak{B}$):

Definition ($\mathscr{S}$-valid)

Given an atomic system $\mathscr{S} \in \mathfrak{B}$:

- Every $\mathscr{S}$-derivation $\mathcal{A}$ is valid
- A closed, normal argument $\mathcal{A}$ is $\mathscr{S}$-valid if its immediate sub-derivation is $\mathscr{S}$-valid
- A closed, non-normal argument $\mathcal{A}$ is $\mathscr{S}$-valid if it reduces to an $\mathscr{S}$-valid argument
- An open argument $\mathcal{A}$ is $\mathscr{S}$-valid iff, for any $\mathscr{S}' \supseteq \mathscr{S}$, the result of extending the argument $\mathcal{A}$ with $\mathscr{S}'$-valid arguments for all the open hypotheses results in an $\mathscr{S}'$-valid argument.

An argument $\mathcal{A}$ is valid if it is $\mathscr{S}$-valid for every atomic system $\mathscr{S} \in \mathfrak{B}$.

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But why?

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We call the study of such judgments *base-extension semantics* (B-eS).

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Without loss of generality, a $\mathscr{B}$-valid argument $\mathcal{A}$ for $\varphi \vee \psi$ is closed and in normal form and concludes with $\vee_{\mathsf{I}}$.

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Therefore, we have the following:

$$\Vdash_{\mathscr{B}} \varphi \vee \psi \qquad \text{iff} \qquad \Vdash_{\mathscr{B}} \varphi \text{ or } \Vdash_{\mathscr{B}} \psi$$

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$$\varphi \Vdash_{\mathscr{B}} \psi \qquad \text{iff} \qquad \forall \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}, \text{ if } \Vdash_{\mathscr{C}} \varphi, \text{ then } \Vdash_{\mathscr{C}} \psi$$

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In general, for $\Delta \neq \varnothing$,

$$\Delta \Vdash_{\mathscr{B}} \psi \qquad \text{iff} \qquad \forall \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}, \text{ if } \Vdash_{\mathscr{C}} \varphi \text{ for } \varphi \in \Delta, \text{ then } \Vdash_{\mathscr{C}} \psi$$

A semantics of logical signs

Definition (Support)

Fix a basis $\mathfrak{B}$ and let $\mathscr{B} \in \mathfrak{B}$ be a base:

$\Vdash_{\mathscr{B}} P$	iff	$\vdash_{\mathscr{B}} P$	(At)
$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$	($\wedge$)
$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ or $\Vdash_{\mathscr{B}} \psi$	($\vee$)
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} P$ for any atom P	($\bot$)
$\Vdash_{\mathscr{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathscr{B}} \psi$	($\to$)
$\Vdash_{\mathscr{B}} \forall x \varphi$	iff	$\Vdash_{\mathscr{B}} \varphi[x \mapsto a]$ for any $a \in \mathbb{C}$	($\forall$)
$\Delta \Vdash_{\mathscr{B}} \varphi$	iff	$\forall \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}$, if $\Vdash_{\mathscr{C}} \psi$ for all $\psi \in \Delta$, then $\Vdash_{\mathscr{C}} \varphi$	(Inf)
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Incompleteness

Theorem (Piecha et al. 2015; 2016; 2019)

Regardless of the basis, the semantics validates the Kreisel-Putnam axiom:

$$\big(\neg\chi \to (\varphi \vee \psi)\big) \to \big((\neg\chi \to \varphi) \vee (\neg\chi \to \psi)\big)$$

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Note that the closely-related Harrop's rule is admissible but not derivable in intuitionistic logic,

$$\frac{\neg\chi \to (\varphi \vee \psi)}{(\neg\chi \to \varphi) \vee (\neg\chi \to \psi)}$$

Theorem (Completeness of the Disjunction-free Fragment)

If $\vee$ *does not occur in* φ*, then* $\Vdash \varphi$ *iff* $\vdash_{\mathsf{NJ}} \varphi$

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So can we axiomatize (naïve) base-extension semantics?

General Inquisitive Logic

Definition (General Inquisitive Logic)

The logic that extends NJ with the generalized Kreisel-Putnam axiom — for all disjunction-free χ, the formula

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If $\mathfrak{B} = \mathfrak{B}_2$ (that is, comprises all second-level atomic systems), then $\Vdash \varphi$ iff φ can be proved in general inquisitive logic.

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The case where $\mathfrak{B} = \mathfrak{B}_1$ (comprises first-level atomic systems) is called Stafford's logic [1].

[1] Don't ask. The axiomatization is not pretty.

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Example

Asserting 'It is raining' enables you to settle 'Is it raining?'. But asking 'Is it raining?' does not suffice to assert 'It is raining'.

We defer to Ciardelli et al. *Inquisitive Semantics* (2018) for more detail.

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In *Base-extension Semantics for Intuitionistic Sentential Logic* (2015), Sandqvist proposed to replace

$$\Vdash_{\mathscr{B}} \varphi \vee \psi \qquad \text{iff} \qquad \Vdash_{\mathscr{B}} \varphi \text{ or } \Vdash_{\mathscr{B}} \psi$$

with

$$\Vdash_{\mathscr{B}} \varphi \vee \psi \qquad \text{iff} \qquad \forall \mathscr{C} \supseteq \mathscr{B}, \forall P, \text{ if } \varphi \Vdash_{\mathscr{C}} P \text{ and } \psi \Vdash_{\mathscr{C}} P, \text{ then } \Vdash_{\mathscr{C}} P$$

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Sandqvist 2015 doesn't really explain where

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It recalls the 'second-order definition' of disjunction by Prawitz:

$$\varphi \vee \psi := \forall \chi \big((\varphi \to \chi) \to (\psi \to \chi) \to \chi \big)$$

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Type Theory

This definition of disjunction is actually typical in type theory, but unusual in logic.

An alternative semantics of logical signs

Definition (Support)

Fix a basis $\mathfrak{B}$ and let $\mathscr{B} \in \mathfrak{B}$ be a base:

$\Vdash_{\mathscr{B}} P$	iff	$\vdash_{\mathscr{B}} P$	(At)
$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$	($\wedge$)
$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	$\forall \mathscr{C} \supseteq \mathscr{B}, \forall P$, if $\varphi \Vdash_{\mathscr{C}} P$ and $\psi \Vdash_{\mathscr{C}} P$, then $\Vdash_{\mathscr{C}} P$	($\vee$)
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} P$ for any atom P	($\bot$)
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The proof of soundness — that is, if φ is provable in intuitionistic logic, then $\Vdash\varphi$ — is standard[2]. The proof of completeness is quite special.

[2]Interestingly, the soundness proof highly resembles that of the so-called 'instantiation overflow' property in F_{at}, the second-order logic with atomic polymorphism.

The Simulation Proof

Theorem (Sandqvist 2015)

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Sandqvist says this about his proof:

> *The mathematical resources required for the purpose are quite elementary; there will be no need to invoke canonical models, König's lemma, or even bar induction. The proof proceeds, instead, by simulating an intuitionistic deduction using basic sentences within a base specifically tailored to the inference at hand.*

Flattening

To each sub-formula χ of φ associate an atom $\chi^\flat$ with the caveat that $\chi^\flat = \chi$ when χ is an atom.

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Construct a special base $\mathscr{N}$ for φ simulating NJ over its subformulae; that is, $\mathscr{N}$ contains all instances of the following rules, where well-defined, as α and β range over subformulae of φ and P is any atom:

$$\frac{[\alpha^\flat]}{\beta^\flat}\,(\alpha \to \beta)^\flat \qquad \frac{\alpha^\flat \quad \beta^\flat}{(\alpha \wedge \beta)^\flat} \qquad \frac{\alpha^\flat}{(\alpha \vee \beta)^\flat} \qquad \frac{\beta^\flat}{(\alpha \vee \beta)^\flat}$$

$$\frac{(\alpha \to \beta)^\flat \quad \alpha^\flat}{\beta^\flat} \qquad \frac{(\alpha \wedge \beta)^\flat}{\alpha^\flat} \qquad \frac{(\alpha \wedge \beta)^\flat}{\beta^\flat} \qquad \frac{(\alpha \wedge \beta)^\flat \quad \begin{matrix}[\alpha^\flat] \\ P\end{matrix} \quad \begin{matrix}[\beta^\flat] \\ P\end{matrix}}{P} \qquad \frac{\bot^\flat}{P}$$

Proof of Completeness

Lemma (†)

For any $\mathscr{N}' \supseteq \mathscr{N}$,

$$\Vdash_{\mathscr{N}'} \alpha^\flat \qquad \textit{iff} \qquad \Vdash_{\mathscr{N}'} \alpha$$

Lemma (‡)

For any P,

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Grigori Mints

A saying in mathematics: *all theorems were proved in the Soviet Union.*

A saying in logic: *all theorems in logic were proved in the Soviet Union by Grigori Mints.*

It turns out, Sandqvist's *flattening* technique is something Mints was already using in the '80s to do proof-search.

Figure: Grigori Mints

Resolution Calculi

Definition (Clausal Systems)

A *clause* is a formula of the following form:

$$(p \to q^*) \to r, \quad p \to (q \vee r), \quad (p_1 \wedge \cdots \wedge p_n) \to q^*,$$

where:

- $p, q, r, p_1, \ldots, p_n$ are basic propositions;
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Due to their structure, there are advantages to doing proof-search relative to clausal systems compared to arbitrary sets of formulae.

Mints' Theorem

Mints gives a construction that systematically takes a formula φ and delivers a clausal system M and atom g with the following property:

Theorem (Mints 1981; 1988)

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This work, along with subsequent work on modal logic, is now part of several automated reasoning engines — for example, *Counterexample-guided Abstraction Refinement* (CEGAR).

Example: Commutativity of Conjunction

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$$a \wedge b \to r, r \to a, r \to b$$
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This looks familiar if we were to write $r := (a \wedge b)^\flat$, $s = (b \wedge a)^\flat$, and $g = \big((a \wedge b) \to (b \wedge a)\big)^\flat$.

Bases and Clausal Systems

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$$[\varepsilon \Rightarrow c] := c,$$
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Theorem (Gheorghiu 2025)

For any base $\mathscr{B}$,

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Figure: Robert Kowalski

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Definition (Goal Clauses, Definite Clauses, Program)

Goal clauses G and *definite clauses* D are defined by the following mutual induction:

$$\begin{array}{lcl} G & := & A \mid D \to G \mid G \wedge G \mid G \vee G \mid \exists x G \\ D & := & A \mid G \to A \mid \forall x D \mid D \wedge D \end{array}$$

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A program $\mathscr{P}$ is a set of definite clauses.

We write $\mathscr{P} \vdash G$ to denote that the goal G may be satisfied by the program $\mathscr{P}$.

Example: Family Tree

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Example (Family)

The following is a program $\mathscr{P}_{\mathsf{family}}$:

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&\texttt{father}(\texttt{Alice}, \texttt{Bob})\\
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This definition is arguably complete until we apply what Hallnäs and Schroeder-Heister called 'definitional reflection' in *A Proof-Theoretic Approach to Logic Programming I, II* (1990, 1991).

Operational Semantics

Miller gives the following operational semantics for his logic programming language:

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\begin{array}{lll}
\mathscr{P} \vdash A & \text{if} & A \in \overline{\mathscr{P}} \qquad \text{(IN)} \\
\mathscr{P} \vdash A & \text{if} & G \to A \in \overline{\mathscr{P}} \text{ and } \mathscr{P} \vdash G \qquad \text{(CLAUSE)} \\
\mathscr{P} \vdash G_1 \vee G_2 & \text{if} & \mathscr{P} \vdash G_1 \text{ or } \mathscr{P} \vdash G_2 \qquad \text{(OR)} \\
\mathscr{P} \vdash G_1 \wedge G_2 & \text{if} & \mathscr{P} \vdash G_1 \text{ and } \mathscr{P} \vdash G_2 \qquad \text{(AND)} \\
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Essentially $\models$ relates this account of logic programming to the standard denotational semantics often used in logic.

Equivalence between LP and B-eS

In *Definite Formulae, Negation-as-Failure, and the Base-extension Semantics of Intuitionistic Propositional Logic* (2024), Gheorghiu and Pym showed that Sandqvist's completeness proof can be read in terms of LP.

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Completeness now corresponds to the following diagram: