Introduction to Proof-theoretic Semantics

Lecture 4: Classical Logic & Kripke Semantics

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Overview

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What have we done?

The story so far:

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- **Lecture 2:** Prawitz gave a notion of 'validity' of proofs based on the idea that Gentzen's introduction rules 'define' their connectives.
- **Lecture 3:** This notion of validity corresponds to general inquisitive logic. Using the elimination rules instead recovers intuitionistic logic.[1]

[1] There is lots of computational content here! And more to explore...

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We considered a certain approach known as *base-extension semantics*.

Base-extension Semantics

Definition (Support)

Fix a basis $\mathfrak{B}$ and let $\mathscr{B} \in \mathfrak{B}$ be a base:

$\Vdash_{\mathscr{B}} P$	iff	$\vdash_{\mathscr{B}} P$	(At)
$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$	($\wedge$)
$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	$\forall \mathscr{C} \supseteq \mathscr{B}, \forall P,$ if $\varphi \Vdash_{\mathscr{C}} P$ and $\psi \Vdash_{\mathscr{C}} P$, then $\Vdash_{\mathscr{C}} P$	($\vee$)
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} P$ for any atom P	($\bot$)
$\Vdash_{\mathscr{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathscr{B}} \psi$	($\to$)
$\Vdash_{\mathscr{B}} \forall x \varphi$	iff	$\Vdash_{\mathscr{B}} \varphi[x \mapsto a]$ for any $a \in \mathbb{C}$	($\forall$)
$\Delta \Vdash_{\mathscr{B}} \varphi$	iff	$\forall \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}$, if $\Vdash_{\mathscr{C}} \psi$ for all $\psi \in \Delta$, then $\Vdash_{\mathscr{C}} \varphi$	(Inf)
$\Gamma \Vdash \varphi$	iff	$\Gamma \Vdash_{\mathscr{B}} \varphi$ for all $\mathscr{B} \in \mathfrak{B}$	

Soundness and Completeness

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It is mathematically elementary and constructive.

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While we have managed to give a proof-theoretic semantics for intuitionistic logic, what about classical logic?

Truth-functional Semantics

Classical logic is typically read in terms of *truth tables*:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
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A	B	$A \rightarrow B$
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This forms a semantics as follows:

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A valuation is a function $v : \mathbb{A} \to \{0, 1\}$ extended to formulae using the truth tables.

Definition (Validity)

$\Gamma \models' \varphi$ iff, for any valuation v, if $v(\psi) = 1$ for $\psi \in \Gamma$, then $v(\varphi) = 1$.

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Definition (Natural Deduction System NK)

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge_{\mathsf{I}} \qquad \frac{\varphi \wedge \psi}{\varphi}\wedge^1_{\mathsf{E}} \quad \frac{\varphi \wedge \psi}{\psi}\wedge^2_{\mathsf{E}} \qquad \frac{\begin{matrix}[\varphi] \\ \psi\end{matrix}}{\varphi \to \psi}\to_{\mathsf{I}} \qquad \frac{\bot}{\varphi}\ \mathsf{EFQ}$$

$$\frac{\varphi}{\varphi \vee \psi}\vee^1_{\mathsf{I}} \quad \frac{\psi}{\varphi \vee \psi}\vee^2_{\mathsf{I}} \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}\vee_{\mathsf{E}} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\to_{\mathsf{E}} \qquad \boxed{\frac{\begin{matrix}[\neg\varphi] \\ \bot\end{matrix}}{\varphi}\ \mathsf{RAA}}$$

Classical logic without Bivalence

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In *Classical logic without Bivalence* (2009)[2], Sandqvist announced that if we restrict to $\mathfrak{B}_1$, then support ($\Vdash$) defines classical logic.

[2] A digested result from his 2005 thesis *An Inferentialist Interpretation of Classical Logic (Uppsala University)*

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Open Problem

Why $\mathfrak{B}_1$ expresses classicality and $\mathfrak{B}_2$ expresses intuitionism is unknown and bewildering.

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Definition (Maxiconsistent Base)

A base $\bar{\mathscr{B}}$ is maxiconsistent iff it is consistent — that is, for some P, it is not the case that $\vdash_{\bar{\mathscr{B}}} P$ — and it is maximal with respect to set-inclusion with this property — that is, any $\mathscr{C} \supsetneq \bar{\mathscr{B}}$ is inconsistent.

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Lemma (Makinson 2014)

If $\nVdash_{\mathscr{B}} A$, then there is a maxiconsistent base $\bar{\mathscr{B}} \supseteq \mathscr{B}$ such that $\nVdash_{\bar{\mathscr{B}}} A$.

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Observe $\nvdash_{\mathscr{M}} B$. Therefore, there is $\bar{\mathscr{M}} \supseteq \mathscr{M}$ maxiconsistent such that $\nvdash_{\bar{\mathscr{M}}} B$. It is easy to see that $\vdash_{\mathscr{M}} P$ iff $\vdash_{\bar{\mathscr{M}}} P$ for any $P \in \mathbb{A}$.

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- If something is necessary, then it is the case — $\Box P \rightarrow P$ (T)
- If something is necessary, then it is necessarily necessary — $\Box P \rightarrow \Box\Box P$ (4)

Modal Logic

A well-studied area of logic is *modal* logic.

A modal is an expression that is used to qualify a judgement (e.g., 'necessarily true', 'ought to be true', and so on).
In logic, we usually use $\Box$ and $\Diamond$ to denote modalities read 'necessarily' and 'possibly'.

We may make different assumptions about the behaviour of these modalities:

- If something is necessary, then it is the case — $\Box P \to P$ (T)
- If something is necessary, then it is necessarily necessary — $\Box P \to \Box\Box P$ (4)
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As to which of these axioms to accept, seek spiritual guidance from your nearest philosopher.

Kripke Semantics

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Definition (Model)

A model is a tuple $\langle \mathbb{W}, R, I \rangle$ in which $\mathbb{W}$ is a set (of worlds), R is a binary relation on $\mathbb{W}$ and $I : \mathbb{A} \to \wp(\mathbb{W})$.

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Satisfaction is defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, w \models A & \text{iff} & w \in I(A) \\
\mathfrak{M}, w \models \bot & & \text{never} \\
\mathfrak{M}, w \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Box\psi & \text{iff} & \text{for any } v\text{, if } Rwv\text{, then } \mathfrak{M}, v \models \psi \\
\mathfrak{M}, w \models \Diamond\psi & \text{iff} & \text{for some } v\text{, } Rwv \text{ and } \mathfrak{M}, v \models \psi
\end{array}$$

Frame Conditions

We define validity by satisfaction in all models:

$$\models \varphi \qquad \text{iff} \qquad \mathfrak{M} \models \varphi \text{ for all models } \mathfrak{M}$$

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These logics always satisfy the following distributive condition (K):

$$\Box(P \to Q) \to (\Box P \to \Box Q)$$

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Definition (Validity)

A formula φ is valid relative to a context Γ iff every model of Γ is a model of φ — that is,

$$\Gamma \models \varphi \quad \text{iff} \quad \text{for any intuitionistic model } \mathfrak{M}, w, \text{ if } \mathfrak{M}, w \models \psi \text{ for } \psi \in \Gamma, \text{ then } \mathfrak{M}, w \models \varphi$$

Intuitionistic Satisfaction

Kripke uses the following clauses for the semantics:

$\mathfrak{M}, w \models P$	iff	$P \in I(w)$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \bot$		never
$\mathfrak{M}, w \models \varphi \to \psi$	**iff**	**for any v, if Rwv and $\mathfrak{M}, v \models \varphi$, then $\mathfrak{M}, v \models \psi$**

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Compare this to Piecha et al.'s semantics:

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\Vdash_{\mathscr{B}} P & \text{iff} & \vdash_{\mathscr{B}} P \\
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\end{array}
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Despite their similarities, intuitionistic logic is **complete** with respect to the former but **not** the latter.

Alternative Kripke Semantics

Recall also that Sandqvist 2015 gave a complete semantics by modifying disjunction:

$$\Vdash_{\mathscr{B}} \varphi \vee \psi \quad \text{iff} \quad \text{for any } \mathscr{C} \text{ such that } \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}\text{, and any atom } P\text{,}$$
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What if we used these clauses in the Kripke semantics? That is, we define:

$$\begin{array}{lll}
\mathfrak{M}, w \models' P & \text{iff} & P \in I(w) \\
\mathfrak{M}, w \models' \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models' \varphi \text{ and } \mathfrak{M}, w \models' \psi \\
\mathfrak{M}, w \models' \varphi \vee \psi & \text{iff} & \textbf{for any } v \textbf{ such that } Rwv\textbf{, and any atom } P, \\
 & & \mathfrak{M}, v \models' (\varphi \to P) \to (\psi \to P) \to P \\
\mathfrak{M}, w \models' \bot & \text{iff} & \mathfrak{M}, w \models' P \textbf{ for any atom } P \\
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Does this now determine an alternative Kripke semantics for intuitionistic logic?

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Theorem

$\Gamma \models' \varphi$ *iff* $\Gamma \Vdash \varphi$.

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Proof.

(LHS $\Rightarrow$ RHS) The idea is that bases form an intuitionistic model, $\mathfrak{S} := \langle \mathfrak{B}, \subseteq_{\mathfrak{B}}, I \rangle$, where $I(\mathscr{B}) := \{P \in \mathbb{A} \mid \Vdash_{\mathscr{B}} P\}$. Completeness follows easily:

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\end{array}
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(RHS $\Rightarrow$ LHS). We know $\Gamma \Vdash \varphi$ implies that there is an NJ-derivation of φ from Γ. Using standard techniques for soundness, it is routine to show that if there is an NJ-derivation of φ from Γ, then $\Gamma \models' \varphi$. $\square$

Local Divergence

So $\models'$ and $\models$ are globally equivalent — that is, $\Gamma \models' \varphi$ iff $\Gamma \models \varphi$.

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Proof.

Ta-da!

$$\begin{array}{ccccc} u & \longleftarrow & w & \longrightarrow & v \\ \{p\} & & \varnothing & & \{q\} \end{array}$$

That is, $\mathfrak{M} := \langle \{w, u, v\}, \{(w, u), (w, v)\}, I \rangle$, where $I(p) = \{u\}$, $I(q) = \{v\}$, $I(r) = \varnothing$ for $r \notin \{p, q\}$. We leave it as an exercise to the reader to show that $\mathfrak{M}, w \not\models p \vee q$ and $\mathfrak{M}, w \models' p \vee q$. $\square$