Introduction to Proof-theoretic Semantics

Lecture 5: Substructural Logic

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Overview

1. Substructural Logics

2. Case Study: Intuitionistic Linear Logic

3. Proof-theoretic Semantics for IMLL

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How do vending machines think?

Abstract Vending Machine

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- The vending machine is programmed such that when given 1€, it outputs a KitKat
- I give the vending machine my 1€ coin, this rule executes and I receive a KitKat

Of course, this can be made more complex — for example, there may be another item (e.g., a packet of Haribo) and I have to choose which one I want.

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Economic Nightmare

The problem is that I can prove that one coin suffices for as many KitKats as I want — for example,

$$\dfrac{\dfrac{\dfrac{[C] \quad C \to K}{K} \qquad \dfrac{[C] \quad C \to K}{K}}{K \wedge K}}{C \to K \wedge K}$$

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- Second, this affects the meaning of conjunction $\wedge$ — if we have only one coin C in the multiset of hypotheses, then

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is not possible. So we need to clarify how the rule governing the connective works in the new setup with multisets.
- Third, the $\rightarrow$ is not what we intend. We should only be able to use the coin *one* time — observe that we discharge C twice when deriving $C \rightarrow K \wedge K$ in the Economic Nightmare, but these are actually several different C in our wallet (i.e., the multiset of hypotheses).

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Substructural Logic

This motivates a new type of logic where we restrict the 'structural' power of the connectives — that is, strictly monitor how they interact and affect the hypotheses.
These are called *substructural* logics.

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We shall presently consider the *intuitionistic* version of this logic.

It comes in three parts: the additives, the multiplicatives, and the 'bang' modality !.

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For the new connectives, we define them as follows where Γ and Δ are multisets of formulae:

$$\dfrac{\Gamma \rhd \varphi \quad \Delta \rhd \psi}{\Gamma \fatsemi \Delta \rhd \varphi \otimes \psi} \qquad \text{and} \qquad \dfrac{\varphi \fatsemi \Gamma \rhd \psi}{\Gamma \rhd \varphi \multimap \psi}$$

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Observe that $\varphi \mathbin{⨾} \Gamma := [\varphi] \sqcup \Gamma$ *has exactly one more occurrence of* φ *than* Γ.

The ‘additive’ fragment of ILL

Definition (Natural Deduction Rules for the Additive Connectives)

We have the additive conjunction (&), the additive disjunction ($\oplus$), the unit of $\oplus$ (0).

$$\frac{\mathit{\Gamma} \rhd \varphi \quad \mathit{\Gamma} \rhd \psi}{\mathit{\Gamma} \rhd \varphi \& \psi}\ \&_{\mathsf{I}} \qquad \frac{\mathit{\Gamma} \rhd \varphi \& \psi}{\mathit{\Gamma} \rhd \varphi}\ \&_{\mathsf{E1}} \qquad \frac{\mathit{\Gamma} \rhd \varphi \& \psi}{\mathit{\Gamma} \rhd \psi}\ \&_{\mathsf{E2}}$$

$$\frac{\mathit{\Gamma} \rhd \varphi}{\mathit{\Gamma} \rhd \varphi \oplus \psi}\ \oplus_{\mathsf{I1}} \qquad \frac{\mathit{\Gamma} \rhd \psi}{\mathit{\Gamma} \rhd \varphi \oplus \psi}\ \oplus_{\mathsf{I1}}$$

$$\frac{\mathit{\Gamma} \rhd \varphi \oplus \psi \quad \mathit{\Delta} \fatsemi \varphi \rhd \chi \quad \mathit{\Delta} \fatsemi \psi \rhd \chi}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \rhd \chi}\ \oplus_{\mathsf{E}} \qquad \frac{\mathit{\Gamma} \rhd 0}{\mathit{\Gamma} \rhd \chi}\ 0_{\mathsf{E}}$$

The 'multiplicative' fragment of ILL

Definition (Natural Deduction Rules for the Multiplicative Connectives)

We have the multiplicative conjunction ($\otimes$), the unit (I), and the multiplicative implication ($\multimap$).

$$\frac{\Gamma \rhd \varphi \quad \Delta \rhd \psi}{\Gamma \mathbin{;} \Delta \rhd \varphi \otimes \psi}\ \otimes_{\mathsf{I}} \qquad \frac{\Gamma \rhd \varphi \otimes \psi \quad \Delta \mathbin{;} \varphi \mathbin{;} \psi \rhd \chi}{\Gamma \mathbin{;} \Delta \rhd \chi}\ \otimes_{\mathsf{E}}$$

$$\frac{}{\varnothing \rhd \mathrm{I}}\ \mathrm{I}_{\mathsf{I}} \qquad \frac{\Gamma \rhd \varphi \quad \Delta \rhd \mathrm{I}}{\Gamma \mathbin{;} \Delta \rhd \varphi}\ \mathrm{I}_{\mathsf{E}}$$

$$\frac{\Gamma \mathbin{;} \varphi \rhd \psi}{\Gamma \rhd \varphi \multimap \psi}\ \multimap_{\mathsf{I}} \qquad \frac{\Gamma \rhd \varphi \multimap \psi \quad \Delta \rhd \varphi}{\Gamma \mathbin{;} \Delta \rhd \psi}\ \multimap_{\mathsf{E}}$$

The ‘bang’ modality of ILL

Definition (Natural Deduction Rules for the Modalities and the Initial Rules)

The ‘bang’ operator enables application of weakening and contraction.

$$\frac{\Gamma \rhd \chi}{\Gamma \,{}_{{}^{\mathfrak{g}}}\, !\varphi \rhd \chi} \text{ Weakening} \qquad \frac{\Gamma \,{}_{{}^{\mathfrak{g}}}\, !\varphi \,{}_{{}^{\mathfrak{g}}}\, !\varphi \rhd \chi}{\Gamma \,{}_{{}^{\mathfrak{g}}}\, !\varphi \rhd \chi} \text{ Contraction}$$

$$\frac{\Gamma \,{}_{{}^{\mathfrak{g}}}\, \varphi \rhd \chi}{\Gamma \,{}_{{}^{\mathfrak{g}}}\, !\varphi \rhd \chi} \text{ Dereliction} \qquad \frac{!\Gamma \rhd \varphi}{!\Gamma \rhd !\varphi} \text{ Promotion}$$

$$\frac{}{\varphi \rhd \varphi} \text{ Initial}$$

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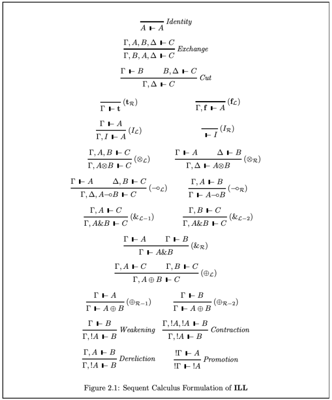

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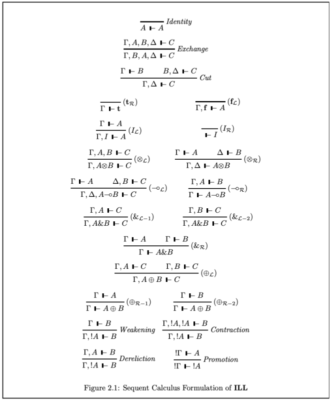

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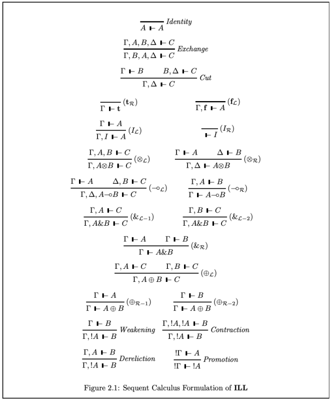
To begin, we will recall the B-eS for intuitionistic logic (see Lecture 3).



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$$\frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}\ (\&_{\mathcal{L}-1}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}\ (\&_{\mathcal{L}-2})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&_{\mathcal{R}})$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus_{\mathcal{R}-2})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textit{Weakening} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textit{Contraction}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \textit{Dereliction} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \textit{Promotion}$$

Figure 2.1: Sequent Calculus Formulation of **ILL**

Figure: G. Bierman, *On Intuitionistic Linear Logic* (1993)

Bases for Intuitionistic Logic

Definition (Base)

A base $\mathscr{B}$ is a set of atomic rules of the form

$$\frac{}{p}\ \mathsf{A} \qquad \frac{\begin{matrix}[P_1] & & [P_n] \\ p_1 & \ldots & p_n\end{matrix}}{p}\ \mathsf{R}$$

— the $P_1, \ldots, P_n$ denote (possibly empty) sets of atomic propositions that may be discharged when R is used. We write $\vdash_{\mathscr{B}} p$ to denote that there is a $\mathscr{B}$-derivation concluding p.

This forms the grounds for defining the logical constants.

Base-extension Semantics for Intuitionistic Logic

Definition (Support)

Fix a basis $\mathfrak{B}$ and let $\mathscr{B} \in \mathfrak{B}$ be a base:

$\Vdash_{\mathscr{B}} P$	iff	$\vdash_{\mathscr{B}} P$	(At)
$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$	($\wedge$)
$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	$\forall \mathscr{C} \supseteq \mathscr{B}, \forall P,$ if $\varphi \Vdash_{\mathscr{C}} P$ and $\psi \Vdash_{\mathscr{C}} P$, then $\Vdash_{\mathscr{C}} P$	($\vee$)
$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} P$ for any atom P	($\bot$)
$\Vdash_{\mathscr{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathscr{B}} \psi$	($\to$)
$\Vdash_{\mathscr{B}} \forall x \varphi$	iff	$\Vdash_{\mathscr{B}} \varphi[x \mapsto a]$ for any $a \in \mathbb{C}$	($\forall$)
$\Delta \Vdash_{\mathscr{B}} \varphi$	iff	$\forall \mathscr{C} \supseteq_{\mathfrak{B}} \mathscr{B}$, if $\Vdash_{\mathscr{C}} \psi$ for all $\psi \in \Delta$, then $\Vdash_{\mathscr{C}} \varphi$	(Inf)
$\Gamma \Vdash \varphi$	iff	$\Gamma \Vdash_{\mathscr{B}} \varphi$ for all $\mathscr{B} \in \mathfrak{B}$	

Resource-sensitive Base-extension Semantics

How do we make this semantics resource-sensitive? The result would be a candidate for a semantics of IMLL.

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- First, our notion of atomic rule, base, and derivability in a base needs to be made substructural.
- Second, the support relation should carry 'resources' around that track how things are consumed when validating formulae.
- Third, we need to carefully consider what it means to support a multiset of formulae (as opposed to a set of them).

Multiplicative Bases I

The first problem:

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— the $P_1, \dots, P_n$ denote (possibly empty) **multisets** of atomic propositions that may be discharged when R is used. We write $\vdash_{\mathscr{B}} p$ to denote that there is a $\mathscr{B}$-derivation concluding p.

Multiplicative Bases II

Definition (Derivability in a Multiplicative Base)

Derivability in a multiplicative base $\mathscr{B}$ — denoted $\vdash_{\mathscr{B}}$ — is defined inductively as follows:

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  then $\vdash_{\mathscr{B}} p$.
- For any multisets of atoms $Q_1, \ldots, Q_n$, if $\mathscr{B}$ has the rule

  $$\frac{\begin{array}{c}[P_1]\\ p_1\end{array} \quad \ldots \quad \begin{array}{c}[P_n]\\ p_n\end{array}}{p}\ \mathrm{R}$$

  and $P_i \fatcomma Q_i \vdash_{\mathscr{B}} p_i$ for $i = 1, \ldots, n$, then $Q_1 \fatcomma \ldots \fatcomma Q_n \vdash_{\mathscr{B}} p$.

Example: Multiplicative Bases

Examples

The following is a multiplicative base $\mathscr{V}$ representing the internal mechanism of a vending machine:

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It is precise in that it says, one coin gives exactly one item. Two coins would be too much for one item — that is, we do not have $C \fatcomma C \vdash_{\mathscr{V}} K$ because one coin is left over in the transaction!

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To be able to express more complex scenarios (e.g., returning change, or buying two items at the same time), we require the logical connectives. So let's continue setting up their B-eS.

Atomic Resources

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> *Second, the support relation should carry 'resources' around that track how things are consumed when validating formulae.*

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We enrich it by a multiset of atoms R thought of as the available 'atomic resources',

$$\mathit{\Gamma} \Vdash^{\mathrm{Q}}_{\mathscr{B}} \varphi \qquad \text{iff} \qquad \text{for any } \mathscr{C} \supseteq \mathscr{B} \text{ and any } \mathrm{U}\text{, if } \Vdash^{\mathrm{U}}_{\mathscr{C}} \mathit{\Gamma}\text{, then } \Vdash^{\mathrm{Q},\mathrm{U}}_{\mathscr{C}} \varphi \qquad \text{(Inf)}$$

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This shows how resources are 'used' when going from assumptions to conclusion. It remains to define $\Vdash^{\mathrm{U}}_{\mathscr{C}} \Gamma$ as Γ is now not a set but a *multiset*.

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This can be formalized as follows:

$$\Vdash^{\mathrm{Q}}_{\mathscr{B}} \Gamma \fatcomma \Delta \qquad \text{iff} \qquad \exists \mathrm{U}, \mathrm{V} \text{ s.t. } \mathrm{Q} = (\mathrm{U} \fatcomma \mathrm{V}), \Vdash^{\mathrm{U}}_{\mathscr{B}} \Gamma, \text{and } \Vdash^{\mathrm{V}}_{\mathscr{B}} \Delta \qquad (\fatcomma)$$

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This completes the setup. It only remains to define the logical signs of IMLL.

Base-extension Semantics for IMLL

Definition (Support)

Let $\mathscr{B}$ be a multiplicative base:

$$
\begin{array}{llll}
\Vdash^{Q}_{\mathscr{B}} p & \text{iff} & Q \vdash_{\mathscr{B}} p & \text{(At)} \\
\Vdash^{Q}_{\mathscr{B}} \varphi \otimes \psi & \text{iff} & \text{for any } \mathscr{C} \supseteq \mathscr{B}, \mathrm{U}, \text{ and } p, \text{ if } \varphi \,⨾\, \psi \Vdash^{\mathrm{U}}_{\mathscr{C}} p, \text{ then } \Vdash^{Q ⨾ \mathrm{U}}_{\mathscr{C}} p & (\otimes) \\
\Vdash^{Q}_{\mathscr{B}} \mathrm{I} & \text{iff} & \text{for any } \mathscr{C} \supseteq \mathscr{B}, \mathrm{U}, \text{ and } p, \text{ if } \Vdash^{\mathrm{U}}_{\mathscr{C}} p, \text{ then } \Vdash^{Q ⨾ \mathrm{U}}_{\mathscr{C}} p & (\mathrm{I}) \\
\Vdash^{Q}_{\mathscr{B}} \varphi \multimap \psi & \text{iff} & \varphi \Vdash^{Q}_{\mathscr{B}} \psi & (\multimap) \\
\Vdash^{Q}_{\mathscr{B}} \Gamma \,⨾\, \Delta & \text{iff} & \exists \mathrm{U}, \mathrm{V} \text{ s.t. } Q = (\mathrm{U} ⨾ \mathrm{V}), \Vdash^{\mathrm{U}}_{\mathscr{B}} \Gamma, \text{and } \Vdash^{\mathrm{V}}_{\mathscr{B}} \Delta & (⨾) \\
\Gamma \Vdash^{Q}_{\mathscr{B}} \varphi & \text{iff} & \text{for any } \mathscr{C} \supseteq \mathscr{B} \text{ and any } \mathrm{U}, \text{if } \Vdash^{\mathrm{U}}_{\mathscr{C}} \Gamma, \text{ then } \Vdash^{Q ⨾ \mathrm{U}}_{\mathscr{C}} \varphi & \text{(Inf)}
\end{array}
$$

Soundness and Completeness

Theorem (Gheorghiu, Gu, and Pym (2023))

$\Gamma \Vdash \varphi$ *iff* $\Gamma \vdash_{IMLL} \varphi$

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$$\Vdash^{R}_{\mathscr{B}} !\varphi \qquad \text{iff} \qquad \begin{array}{l} \text{for any } \mathscr{C} \supseteq \mathscr{B}, \text{ atomic multisets } S \text{ and atom } P, \\ \text{if } \Vdash^{\varnothing}_{\mathscr{D}} \varphi \text{ or } \Vdash^{R}_{\mathscr{D}} p \text{ for any } \mathscr{D} \supseteq \mathscr{C}, \text{ then } \Vdash^{R,S}_{\mathscr{C}} p \end{array}$$

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Gheorghiu, Gu, and Pym (2025) have also shown that one can capture another class of substructural logics known as 'bunched' logics by adding another context-former.

Goodbye!

End.